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To: S&EC Programmers

From: Martin Jacobs

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Subject: POLYNOMIAL FACTORIZATION

New Hitchcock's Method Routines

Several new routines for factoring higher order polynomials are described in this memorandum. Using CS II code, the routines employ a method first described by Frank L. Hitchcock (Jour. Math. and Phys. 1944, p. 69) and theoretically can handle polynomials with real coefficients up to 100th degree or higher, subject to certain restrictions.

Roots of the polynomial are printed out as quadratic factors, from which the roots are easily obtained by the quadratic formula. Roots may be real, imaginary, or complex; complex roots of the form A + ib appear with their conjugates, also roots, in the same quadratic.

I. The Method

Hitchcock's method starts with an approximate quadratic divisor

$$\bar{D} = X^2 + \bar{p}X + \bar{q}$$

in the rough neighborhood of a true divisor $X^2 + Px + Q$. Two quantities Δp and Δq are computed. If the trial quadratic is sufficiently close to the true one, the quadratic

$$D_1 = x^2 + (p_0 + \Delta p_0)x + (q_0 + \Delta q_0) = x^2 + p_1 x + q_1$$

is a better approximation than the first one. The process is repeated, with p_n and q_n converging to $\bar{p}$ and $\bar{q}$, respectively. Iteration is halted when the absolute values of $\Delta p/p$ and $\Delta q/q$ are as small as desired, generally 10^{-8} or 10^{-9} for these routines. The polynomial is then divided through by the quadratic and the process repeats, selecting a new trial divisor and using the final quotient as a new polynomial. Eventually, all factors are obtained.

Certain cases have been found for which Hitchcock's method does not converge. Empirically, the following conditions seem to cause the method to fail:

a. Multiplicity of roots or quadratics

b. In some cases, closeness of roots.

c. Zero roots

d. Other as yet unknown causes

II. The Available Routines

A. General

Programmers should generally use fc TAPE 141-227-75 The coefficients p and q of the quadratics are printed out in columns to seven decimal places. At the bottom of the p column is a root product check; under the q column is a root sum check. These are 8-digit normalized numbers; both should equal +1.0000000 to five or six digits. Results will then possess five or six digit accuracy. Actual running time for a sixth or seventh degree polynomial is about two or three minutes; larger polynomials take proportionately longer.

fc TAPE 141-227-74 is a special purpose trouble-shooting tape. In operation it is identical to the programmers' routine above, but, in addition to solutions and root checks, it prints out each trial divisor as it is used, each successive value for p and q, and each successive reduced polynomial.

The routines now on file are (24, 6). For special use the routine may be used for any (30-J, J). The only changes are:

1. the heading

2. the contents of c3, which is a single register positive integer, must equal $2^J - 3$.

In some cases the program may print out "+.00 .00 .00", preceded by no quadratics, or fewer than expected. This indicates that 55 iterations with each trial divisor have not produced the next quadratic. Unless better approximate divisors are available, Hitchcock's method will not solve the polynomial.

Hitchcock's method does not seem to solve cubic equations. Therefore, when an odd-degree polynomial is factored, the last three roots are left as a cubic equation which can be solved algebraically. Its four coefficients are printed in a row in order of descending powers of the variable.

All zero roots are factored out automatically, and are not printed out or otherwise indicated. They are easily found from the original polynomial by algebraic factoring.

B. Special Characteristics

As mentioned above, an initial approximation must be supplied for each quadratic. The routines themselves provide two, by taking the three leading terms

$$a_0 + a_1x + a_2x^2$$

and the three final ones

$$a_{n-2}\, x^{n-2} + a_{n-1}\, x^{n-1} + a_n\, x^n$$

as trial divisors. After each quadratic is extracted, these approximations are computed anew for the reduced polynomial.

Although the machine approximations are usually sufficient, they do not always give convergence. If a programmer can predict rough values for the roots, he should supply them, as many as he likes, in the data tape. If the machine approximations do not work, the programmed divisors will be used.

C. Data input

Data is programmed as below:

(30-J,J)

$$\left.\begin{array}{ll} \text{a1, } A_n & \text{(coefficient of highest power of x)} \\ \quad A_{n-1} & \\ \quad A_{n-2} & \\ \quad \circ & \\ \quad \circ & \\ \quad \circ & \\ \quad \circ & \\ \quad A_1 & \\ \quad A_0 & \text{(constant term)} \end{array}\right\} \text{G.D.}$$

$$\text{p3, } \left.\begin{array}{l} +0.0 \\ +0.0 \\ \bar{p}_1 \\ \bar{p}_2 \\ \circ \\ \circ \\ \circ \\ \bar{p}_{n-2} \end{array}\right\} \text{G.D.} \qquad \text{q3, } \left.\begin{array}{l} +0.0 \\ +0.0 \\ \bar{q}_1 \\ \bar{q}_2 \\ \circ \\ \circ \\ \circ \\ \bar{q}_{n-2} \end{array}\right\} \text{G.D.}$$

k1, n (degree of polynomial)

k2, m (number of trial divisors, including two initial zeroes.)

START AT w1

III. Subroutines

Because of the present lack of convergence information and the unpredictability of this method, preparation of a Hitchcock factorization subroutine is not recommended at present. The author hopes to continue research on this problem, and eventually some Hitchcock subroutine may be available.

Signed: Martin A. Jacobs

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