

# Simplified Voronoi Diagrams

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## Abstract:

We are interested in Voronoi diagrams as a tool in robot path planning, where the search for a path in an  $r$  dimensional space may be simplified to a search on an  $r - 1$  dimensional Voronoi diagram. We define a Voronoi diagram  $V$  based on a measure of distance which is not a true metric. This formulation has lower algebraic complexity than the usual definition, which is a considerable advantage in motion planning problems with many degrees of freedom. In its simplest form, the measure of distance between a point and a polytope is the maximum of the distances of the point from the half-spaces which pass through faces of the polytope. More generally, the measure is defined in configuration spaces which represent rotation. The Voronoi diagram defined using this distance measure is no longer a strong deformation retract of free space, but it has the following useful property: any path through free space which starts and ends on the diagram can be continuously deformed so that it lies entirely on the diagram. Thus it is still complete for motion planning, but it has lower algebraic complexity than a diagram based on the euclidean metric.

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# 1 Introduction

The Voronoi diagram has proved to be a useful tool in a variety of contexts in computational geometry. Our interest here is in using the diagram to simplify the planning of collision-free paths for a robot among obstacles, the so-called generalized movers' problem. The Voronoi diagram, as usually defined, is a *strong deformation retract* of free space so that free space can be continuously deformed onto the diagram. This means that the diagram is complete for path planning, i.e. Searching the original space for paths can be reduced to a search on the diagram. Reducing the dimension of the set to be searched usually reduces the time complexity of the search. Secondly, the diagram leads to robust paths, i.e. paths that are maximally clear of obstacles.

The Voronoi diagram generated by a set of points in a Euclidean space partitions the space into convex regions which have a single nearest point under some (usually  $L_2$ ) metric. A generalized Voronoi diagram can be defined for points and line segments in the plane (Lee and Drysdale, 1981) which partitions the plane into (generally non-convex) regions. In both cases the diagram is defined to be the set of points equidistant from two or more generators under the appropriate metric. This construction has proved to be useful for motion planning among a set of obstacles in configuration space (see Ó'Dúnlaing and Yap (1982), Ó'Dúnlaing, Sharir, and Yap (1984), Yap (1984), and the textbook of Schwartz and Yap (1986) for an introduction and review of the use of Voronoi diagrams in motion planning). Its virtue for motion planning is that the diagram is a strong deformation retract of free space, i.e. the space outside the obstacles can be continuously deformed onto the diagram. To find a path between two points in free space, it suffices to find a path for each point onto the diagram, and to join these points with a path that lies wholly on the diagram.

The simplified diagram has lower algebraic complexity than the  $L_2$  diagram. For example, in  $\mathbb{R}^3$ , the  $L_2$  diagram about polyhedral obstacles consists of quadric sheets; the simplified diagram is piecewise linear. In  $\mathbb{R}^2$ , the simplified diagram for polygonal obstacles is a graph of straight lines, see fig. 1. In general, the simplified diagram has the same degree as the algebraic obstacle constraints. However, it may not have linear size in the worst case.

One useful aspect of the simplified Voronoi diagram is that it is naturally defined for the six dimensional configuration space of an arbitrary 3D polyhedron moving amidst 3D polyhedral obstacles. In general, it has been considered difficult to define such a Voronoi-type diagram in this case. Our definition elaborates a suggestion of Donald (1984) and Canny (1985), who describe certain Voronoi-like properties of the algebraic set  $\bigcup_{i \neq j} \ker(C_i - C_j)$  for a set of algebraic constraints  $\{C_i\}$ .

## 2 Object-Obstacle Constraints

We briefly derive conditions for overlap of two polyhedral objects  $A$  and  $B$ . A more complete derivation of an equivalent condition is given in Canny (86). The form we derive

here is different from (Canny 86) in that it uses a local test for non-overlap, rather than overlap. We assume first that  $A$  and  $B$  are convex, and then generalize to the non-convex case by taking the conjunction of pairwise non-overlap predicates between convex pieces.

The overlap predicates in Canny (86) generate a shallow (depth 2) AND-OR predicate tree, whose root is a disjunction. It will be advantageous to make the predicate tree as deep as possible and it is also desirable for the root to be a conjunction. So instead we use the following test based on conditions for non-overlap.

**Definition.**

For any face  $f$  of a convex polyhedron  $A$ , the *affine hull*  $\bar{f}$  of  $f$  is the plane which contains  $f$ .

The affine hull of a face  $f$  defines two closed half-spaces, one of which contains  $A$ . We call this half-space the *interior half-space*, and denote it  $\bar{f}^-$ .

Finally, we define the *wedge* of an edge  $e$  of  $A$  as the intersection of the two interior half-spaces of the faces which cobound  $e$ . The wedge of  $e$  is denoted  $\hat{e}$ , and it contains  $A$ .

**Lemma.** Two convex polyhedra  $A$  and  $B$  are non-overlapping iff all edges of  $A$  (or  $B$ ) are outside some wedge of an edge of  $B$  (resp.  $A$ ).

**Proof.** This condition is clearly sufficient for non-overlap, by convexity of  $A$  and  $B$ . Conversely, if  $A$  and  $B$  are disjoint, then there is a (not necessarily unique) non-zero shortest vector between them. Let  $p_A$  and  $p_B$  be the end-points of this vector in  $A$  and  $B$  respectively. If one of these points lies in the interior of a face  $f$ , then the test succeeds for any wedge of an edge in the boundary of  $f$ . If one of the points lies in the interior of an edge  $e$ , then the test succeeds for  $\hat{e}$ .

The only case remaining is where  $p_A$  and  $p_B$  are both vertices. Let  $f$  be a face adjacent to  $p_A$ , and such that  $p_B$  lies outside the interior half-space  $\bar{f}^-$ , (there must be at least one such  $f$ , or  $p_B$  would be contained in  $A$ ). Then if  $B$  is also outside this half-space, the test succeeds for the wedge of any edge that cobounds  $f$ .

Otherwise, let  $W_{p_B}$  be the intersection of the wedges of all edges that cobound  $p_B$ . Let  $S$  be the plane passing through  $p_B$  and normal to the vector  $(p_B - p_A)$ .  $S$  defines two closed half-spaces, one ( $S_A$ ) containing  $A$  and the other ( $S_B$ ) containing both  $B$  and  $W_{p_B}$ , i.e.  $S$  separates  $A$  and  $B$ . Then the intersection  $(W_{p_B} \cap \bar{f})$  lies in  $(S_B \cap \bar{f})$ . Let  $p_0$  be the closest point in  $(W_{p_B} \cap \bar{f})$  to  $(S \cap \bar{f})$ . Then  $p_0$  is in the boundary of some wedge  $\hat{e}$  of an edge  $e$  that cobounds  $p_B$ . Now  $(\hat{e} \cap \bar{f}) \subset (S_B \cap \bar{f})$  and so by projection from  $p_B$ ,  $(\hat{e} \cap \bar{f}^-) \subset (S_B \cap \bar{f}^-)$ . But  $A \subset (S_A \cap \bar{f}^-)$ , so  $(\hat{e} \cap A) = \emptyset$ , and the test succeeds for  $\hat{e}$ .  $\square$

Thus we can define the following predicate for non-overlap  $F_{A,B}$  of  $A$  and  $B$  from the above test:

$$F_{A,B} = \left( \bigvee_{\substack{e_j \in \\ \text{edges}(B)}} \bigwedge_{\substack{e_i \in \\ \text{edges}(A)}} F_{\hat{e}_i, \hat{e}_j} \right) \vee \left( \bigvee_{\substack{e_i \in \\ \text{edges}(A)}} \bigwedge_{\substack{e_j \in \\ \text{edges}(B)}} F_{\hat{e}_i, e_j} \right) \quad (1)$$

where  $F_{\hat{e}_i, e_j} = ((\hat{e}_i \cap e_j) = \emptyset)$ . The corresponding condition for overlap is in the form of a conjunction of disjunctions, as desired, and this is the form we will use in our development. If the object consists of several convex pieces  $A_i$ , as do the obstacles  $B_j$ , then the non-overlap predicate is the conjunction of pairwise predicates

$$F = \bigwedge_i \bigwedge_j F_{A_i, B_j} \quad (2)$$

We must now decompose the non-overlap predicate for a wedge  $\hat{e}_A$  of  $A$  and a edge  $e_B$  of  $B$  into simple geometric predicates that can be computed directly. These geometric predicates are  $A_{f,p}$ ,  $C_{e_A, e_B}^+$  and  $C_{e_A, e_B}^-$ .  $A_{f,p}$  indicates non-overlap of a vertex  $p$  of  $B$  and the interior half-space of a face  $f$  of  $A$  and is given by

$$A_{f,p} = (\mathbf{n}_f \cdot p - c_f > 0) \quad (3)$$

where  $\mathbf{n}_f$  is the outward normal of  $f$ , and  $c_f$  is its distance from the origin. For  $C_{e_A, e_B}^+$  and  $C_{e_A, e_B}^-$  we need the following definitions: Let  $d_A$  and  $d_B$  be the vector directions of  $e_A$  and  $e_B$  respectively, and let  $p_A$  be any point on  $e_A$ . Let  $H$  and  $T$  be the head and tail vertex respectively of  $e_B$ , then we have

$$C_{e_A, e_B}^+ = (H - p_A) \cdot (d_A \times d_B) > 0 \quad (4)$$

$$C_{e_A, e_B}^- = (H - p_A) \cdot (d_A \times d_B) < 0$$

Now we can define  $F_{\hat{e}_A, e_B}$  in terms of the above predicates, and  $L$  and  $R$  which are the left and right faces respectively, which cobound  $e_B$ , (left and right are determined here by viewing  $e_B$  from outside  $\hat{e}_B$  with  $d_B$  upward).

$$F_{\hat{e}_A, e_B} = ((A_{L,H} \wedge (A_{L,T} \vee (A_{R,T} \wedge C_{e_A, e_B}^+))) \vee (A_{R,H} \wedge (A_{R,T} \vee (A_{L,T} \wedge C_{e_A, e_B}^-)))) \quad (5)$$

The predicate  $F_{\hat{e}_A, e_B}$  can be written in the equivalent form:

$$F_{\hat{e}_A, e_B} = (A_{L,H} \vee A_{R,H}) \wedge (A_{L,T} \vee A_{R,T}) \wedge (A_{L,H} \vee A_{R,T} \vee C_{e_A, e_B}^+) \wedge (A_{R,H} \vee A_{L,T} \vee C_{e_A, e_B}^-) \quad (6)$$

There is a similar form for the predicate  $F_{e_A, \hat{e}_B}$ . From (1), (2) and (6) we see that the overall form of the non-overlap predicate can be written as

$$F = \bigwedge_i \bigvee_j \bigwedge_k \bigvee_l (C_{ijkl} > 0) \quad (7)$$

### 3 The Voronoi Diagram

For motion planning, the configuration of object  $A$  is variable. For a polyhedron in three dimensional space, the configuration contains a position  $\mathbf{x} \in \mathbb{R}^3$  and an orientation  $\mathbf{q} \in SO(3)$  component, where  $SO(3)$  is the group of three-dimensional rotations. Thus the face normals, vertex locations, and edge directions of  $A$  are all functions of  $\mathbf{x}$  and  $\mathbf{q}$ . The predicate (7) is now also a function of the configuration  $(\mathbf{x}, \mathbf{q})$ , i.e.

$$F(\mathbf{x}, \mathbf{q}) = \bigwedge_i \bigvee_j \bigwedge_k \bigvee_l (C_{ijkl}(\mathbf{x}, \mathbf{q}) > 0) \quad (8)$$

The forms of the functions  $C_{ijkl}(\mathbf{x}, \mathbf{q})$  are given explicitly in Canny (1986), and they are algebraic if, say, a quaternion representation of rotation is used. The set of overlap configurations is called the *configuration obstacle* and is denoted  $CO = \{(\mathbf{x}, \mathbf{q}) | \neg F(\mathbf{x}, \mathbf{q})\}$ . It may be thought of as a physical obstacle in configuration space to be avoided by a path planner. We now observe that by letting positive real values represent logical one, and non-positive values represent logical zero, that the min function implements logical AND, and the max function implements logical OR. Thus an equivalent form to (8) is

$$F(\mathbf{x}, \mathbf{q}) = \left( \left( \min_i \left( \max_j \left( \min_k \left( \max_l C_{ijkl}(\mathbf{x}, \mathbf{q}) \right) \right) \right) \right) \right) > 0 \quad (9)$$

which suggests that the quantity

$$\rho(\mathbf{x}, \mathbf{q}) = \min_i \left( \max_j \left( \min_k \left( \max_l C_{ijkl}(\mathbf{x}, \mathbf{q}) \right) \right) \right) \quad (10)$$

can be used as a measure of distance from the configuration obstacle, because it varies continuously through configuration space, is positive at configurations outside  $CO$ , and non-positive at configurations inside  $CO$ . Thus the configuration obstacle can be rewritten as  $CO = \{\mathbf{p} | \rho(\mathbf{p}) \leq 0\}$ , and its complement, the set of points in free space can be written  $F = \{\mathbf{p} | \rho(\mathbf{p}) > 0\}$ .

In order to define the Voronoi diagram under the distance measure  $\rho$ , we need a notion of closest feature. The closest features to a configuration  $(\mathbf{x}, \mathbf{q})$  are those  $C_{ijkl}$  which are *critical* in determining the value of  $\rho(\mathbf{x}, \mathbf{q})$ , that is, small changes in the value of  $C_{ijkl}$  cause identical changes in the value of  $\rho$ .

#### Definition.

A constraint  $C_{i_0 j_0 k_0 l_0} \in \{C_{ijkl}\}$  is *critical* at a configuration  $(\mathbf{x}, \mathbf{q})$  if the value of  $C_{i_0 j_0 k_0 l_0}(\mathbf{x}, \mathbf{q})$  equals the maximum (or minimum) value of every max (resp. min) ancestor of  $C_{i_0 j_0 k_0 l_0}$  in the min-max tree in (10). i.e.

$$C_{i_0 j_0 k_0 l_0}(\mathbf{x}, \mathbf{q}) = \max_l C_{i_0 j_0 k_0 l}(\mathbf{x}, \mathbf{q}) = \min_k \left( \max_l C_{i_0 j_0 k l}(\mathbf{x}, \mathbf{q}) \right) = \dots \quad (11)$$

Now we have

**Definition.**

The *simplified Voronoi diagram*  $V$  is the set of configurations in free space  $F$  at which at least two distinct constraints are critical.

It should be clear from the definition of criticality that  $V$  is semi-algebraic if the constraints  $C_{ijkl}$  are algebraic.  $V$  is closed as a subset of free space, although it is not closed in configuration space. Notice that  $V$  has no interior, since it is contained inside a finite set of bisectors, each of which has no interior. A bisector is the zero set of  $(C_{i_0j_0k_0l_0} - C_{i_1j_1k_1l_1})$  for some pair of distinct constraints. It will prove useful to subdivide the Voronoi diagram into two parts:

**Definition.**

The *concave part* of the Voronoi diagram  $V$  denoted  $\text{conc}(V)$  is the set of configurations in  $F$  where two distinct constraints are critical, and the lowest common ancestor of these constraints in the min-max tree of (10) is a min-node.

The *convex part* of the Voronoi diagram  $V$  denoted  $\text{conv}(V)$  is the set of configurations in  $F$  where two distinct constraints are critical, and the lowest common ancestor of these constraints is a max-node.

Notice that these two definitions are not mutually exclusive, because there may be points where more than two constraints are critical, and which satisfy both definitions. Thus  $\text{conv}(V)$  and  $\text{conc}(V)$  may overlap.

## 4 Completeness for Motion Planning

Our key result is that any path in  $F$  with endpoints in  $V$  can be deformed (in  $F$ ) to a path with the same endpoints lying entirely in  $V$ . We start with a path  $p : I \rightarrow \mathbb{R}^3 \times SO(3)$  lying in free space,  $p(I) \subset F$ , where  $I = [0, 1]$  is the unit interval.

First we assume *wlog* that  $p$  intersects  $V$  at a finite number of points. We can do this because, as defined in the previous section,  $V$  is a semi-algebraic set, and by Whitney's (1957) result, it can be split into a finite number of manifolds, or *strata*. Since  $V$  has no interior, all these manifolds have codimension at least one. For *any* path  $p$  there is a path  $p'$  arbitrarily close to  $p$ , which is an embedding of  $I$ , and so  $p'(I)$  is a 1-manifold. Almost every perturbation of  $p'$  intersects all of the strata transversally, and therefore at a finite number of points. We can choose such a perturbation to be arbitrarily small, in particular, smaller than the minimum distance from  $p'(I)$  to  $CO$ . Such a perturbation gives a new path  $p''$  which is path homotopic to  $p$ , and which has finite intersection with  $V$ .

So we assume that  $p$  has  $m$  intersections with  $V$ , and that these occur at points  $p(x_i)$  with  $x_1, \dots, x_m \in I$ , and  $x_1 = 0$  and  $x_m = 1$ . We then break each interval  $[x_i, x_{i+1}]$  in half, giving us two intervals sharing an endpoint. Thus we now have  $2m-2$  intervals each of which intersects  $V$  at only one endpoint. Below we give homotopies for each of these intervals which continuously deform the image of the interval onto  $V$ . Since all these homotopies

agree at their endpoints, they can be pasted together to give us a global homotopy which deforms  $p$  onto  $V$ . For simplicity, we assume the path segment is parametrized in the range  $I = [0, 1]$  and that  $p(0) \in V$ .

The motion constraints  $C_{ijkl}$  are either  $A$  or  $C$  predicates, (3) and (4), and all can be written in the general form below, called the parametrized plane equations by Lozano-Perez (1983).

$$C_{ijkl}(\mathbf{x}, \mathbf{q}) = \mathbf{N}_{ijkl}(\mathbf{q}) \cdot \mathbf{x} + c_{ijkl}(\mathbf{q}) \quad (12)$$

Where  $\mathbf{N}_{ijkl}(\mathbf{q}) \in \mathfrak{R}^3$  and  $c_{ijkl}(\mathbf{q}) \in \mathfrak{R}$  are both continuous functions of  $\mathbf{q}$ . We assume that the  $C_{ijkl}$  are normalized so that  $|\mathbf{N}_{ijkl}(\mathbf{q})| = 1$  for all  $\mathbf{q}$ . Our objective is to continuously deform the path  $p$  onto the diagram, and we use the  $\mathbf{N}_{ijkl}$  as ‘normals’ to push a point on the path  $p(I)$  away from the critical  $C_{ijkl}$ . We assume that the set of positions is bounded by some set of constraint ‘walls’ of the same form as (1), so that a point can be displaced only a finite distance in free space. We also assume that the workspace has unit diameter.

**General position assumptions.** The construction requires the following general position assumptions. First, suppose  $C_{ijkl}$  is type  $C$  predicate (4). Then  $\mathbf{N}_{ijkl}(\mathbf{q}) = d_A(\mathbf{q}) \times d_B$ . To normalize  $\mathbf{N}_{ijkl}$ , we must divide by its magnitude, which must remain non-zero. Hence the set of configurations

$$\{ \mathbf{q} \mid d_A(\mathbf{q}) \times d_B = 0 \} \subset SO(3) \quad (13)$$

must not intersect the image of  $p$ . However, (13) is clearly of codimension 2 in  $SO(3)$ , and hence by Sard’s lemma there is always an arbitrarily small perturbation of  $p$  which avoids (13).

Similarly, the set

$$\{ (\mathbf{x}, \mathbf{q}) \mid (\mathbf{N}_{ijkl}(\mathbf{q}) = \mathbf{N}_{i'j'k'l'}(\mathbf{q})) \wedge (C_{ijkl}(\mathbf{x}, \mathbf{q}) = C_{i'j'k'l'}(\mathbf{x}, \mathbf{q})) \} \quad (14)$$

is also of codimension  $\geq 2$  in  $\mathfrak{R}^3 \times SO(3)$ , and we assume that  $p$  avoids it as well.

We must define two different homotopies depending on whether  $p(0) \in \text{conc}(V)$ , which is the simplest case, or  $p(0) \in \text{conv}(V)$ .

Notice that since  $p(0)$  is the only point on the path which is in  $V$ , there is exactly one constraint which is critical at all configurations in  $p(0, 1]$  (since constraints change value continuously along the path and for another constraint to become critical, it must first equal the first constraint, which can only occur at points on the diagram). Let the critical constraint be  $C_{ijkl}$ . We define a homotopy  $J_0 : I \times I \rightarrow \mathfrak{R}^3 \times SO(3)$  as

$$J_0(t, u) = p(t) + u\mathbf{N}_{ijkl}(\pi_q(p(t))) \quad (15)$$

where  $\pi_q(p(t))$  is the orientation component of  $p(t)$  and the addition symbol means we add the vector quantity  $u\mathbf{N}_{ijkl}(\pi_q(p(t)))$  to the position component of  $p(t)$ . The deformation above pushes points beyond the diagram, so we define a second homotopy  $J_1 : I \times I \rightarrow F$ .

$$J_1(t, u) = J_0(t, \min(u, u_c(t))) \quad (16)$$

$$\text{where } u_c(t) = \inf\{s \mid J_0(t, s) \in \text{conc}(V)\}$$

Recall from the definition of homotopy, that  $J_1$  is a homotopy of  $p$  and a path  $p'$  if  $J_1$  is continuous and  $J_1(t, 0) = p(t)$  and  $J_1(t, 1) = p'(t)$ . The homotopy  $J_1$  suffices to map paths with one endpoint in  $\text{conc}(V)$  onto  $V$ :

**Lemma 4.1** Let  $p : I \rightarrow \mathfrak{R}^3 \times SO(3)$  be a path having  $p(0) \in V$  and no other points in  $V$ . Then  $J_1$  is a homotopy of  $p$  and a path  $p'$  such that  $p'(I) \subset \text{conc}(V)$ . Furthermore  $p'(0) = p(0)$  if  $p(0) \in \text{conc}(V)$ .

**Proof.** From the definition of  $J_1$  we have  $J_1(t, 0) = p(t)$  and  $J_1(t, 1) \in \text{conc}(V)$ . Also if  $p(0) \in \text{conc}(V)$  then  $u_c(0) = 0$  so  $p'(0) = p(0)$ . It remains to show that  $J_1$  is continuous. First we notice that  $J_0$  is continuous. Continuity of  $J_1$  follows if we can show that  $u_c(t)$  is continuous. Now  $J_0(t, u_c(t))$  is contained in the zero sets of all bisectors  $\{(C_{i'j'k'l'} - C_{ijkl})\}$ . Let  $u'_c(t)$  be a deformation onto a *particular* bisector:

$$u'_c(t) = \inf\{u \mid J_0(t, u) \in \ker(C_{i'j'k'l'} - C_{ijkl})\} \quad (17)$$

then  $u'_c$  is continuous because by definition

$$(C_{ijkl} - C_{i'j'k'l'})(J_0(t, u'_c(t))) = 0 \quad (18)$$

and differentiating

$$\frac{\partial}{\partial t}(C_{ijkl} - C_{i'j'k'l'})(J_0(t, u'_c)) + \left(\frac{\partial u'_c}{\partial t}\right) \frac{\partial}{\partial u'_c}(C_{ijkl} - C_{i'j'k'l'})(J_0(t, u'_c)) = 0 \quad (19)$$

rearranging we obtain

$$\frac{\partial u'_c}{\partial t} = -\frac{\frac{\partial}{\partial t}(C_{ijkl} - C_{i'j'k'l'})(J_0(t, u'_c))}{\frac{\partial}{\partial u'_c}(C_{ijkl} - C_{i'j'k'l'})(J_0(t, u'_c))} \quad (20)$$

and therefore  $\frac{\partial}{\partial t}u'_c(t)$  is finite, because the denominator above is non-zero by our general position assumption. Now we observe that  $u_c(t)$  can be constructed by pasting together segments of  $u'_c(t)$  for various bisectors  $\in \text{conc}(V)$ , and we must show that they agree at their endpoints. The proof is by contradiction. Suppose we had

$$u_c(t) = \begin{cases} u'_c(t) & \text{for } t \in (t_0, t_1]; \\ u''_c(t) & \text{for } t \in (t_1, t_2); \end{cases} \quad (21)$$

$$\text{with } u'_c(t_1) \neq u''_c(t_1)$$

for  $u'$  and  $u''$  derived from distinct bisectors. Then since  $u_c$  is the minimum of all such  $u$ , we must have  $u'_c(t_1) < u''_c(t_1)$



But this means that as  $u$  increases,  $J_0(t_1, u)$  crosses two bisectors in  $\text{conc}(V)$  between  $C_{ijkl}$  and other constraints. This is impossible because all constraints  $C$  have  $|\mathbf{N}(\mathbf{q})| = 1$ , and it follows that

$$\frac{\partial}{\partial u} C_{i'j'k'l'}(J_0(t_1, u)) \leq 1 \quad (22)$$

$$\frac{\partial}{\partial u} C_{ijkl}(J_0(t_1, u)) = 1$$

That is, all constraints increase no faster than  $C_{ijkl}$  with the deformation parameter  $u$ . By our general position assumption if  $C_{ijkl} \neq C_{i'j'k'l'}$ , then the inequality in (22) is strict, and so if the clause  $C_{i'j'k'l'}$  becomes critical at  $u'_c$ , it shares a min node lowest ancestor with  $C_{ijkl}$ . Since all constraints in the tree increase more slowly than  $C_{ijkl}$ , the value of this min node will be less than the value of  $C_{ijkl}$  for  $u > u'_c(t_1)$ . Therefore  $C_{ijkl}$  cannot be critical for any  $u > u'_c(t_1)$ , contradicting the assumption that  $u''_c(t_1)$  is distinct from  $u'_c(t_1)$ . So all bisectors in  $\text{conc}(V)$  between  $C_{ijkl}$  and other constraints agree at their endpoints, and  $u_c(t)$  is continuous by pasting. This shows that  $J_1$  is continuous.  $\square$

If  $p(0) \in \text{conv}(V)$  the situation is more complicated, because the deformation  $J_1$  pushes  $p(0, 1]$  away from  $p(0)$ . To correct this, we first compress the first half of  $p$  to a point:

**Lemma 4.2** Let  $p : I \rightarrow F$  be a path in free space. Then  $p$  is homotopic to a path  $p'$  such that

- (i)  $p'([0, \frac{1}{2}]) = p(0)$
- (ii)  $p'(1) = p(1)$
- (iii)  $p'(I) = p(I)$

**Proof.** The required homotopy is

$$J(t, u) = \begin{cases} p(0) & \text{if } t \leq \frac{1}{2}u; \\ p\left(\frac{2t-u}{2-u}\right) & \text{otherwise;} \end{cases} \quad (23)$$

so  $J(t, 0) = p(t)$ , and  $J(t, 1) = p'(t)$ .  $\square$

We apply the homotopy of lemma 4.2 to  $p$  to give us a path  $p'$ . Applying the homotopy  $J_1$  to the path segment  $p'|_{[\frac{1}{2}, 1]}$  continuously deforms this path segment onto  $\text{conc}(V)$ . Then we define a new homotopy which slowly “unravels”  $p'|_{[0, \frac{1}{2}]}$  from  $p(0)$ . All the points in this homotopy have the same orientation, and for each value of the deformation parameter  $u$ , the path consists of a finite number of straight line segments.

The construction is inductive and we start with a homotopy that gives us two straight line segments. If the orientation at the configuration  $p(0)$  is  $\mathbf{q}_0$ , every point on the joining

path will also have orientation  $\mathbf{q}_0$ . We define two vectors in position space  $\mathbf{N}_0$  and  $\mathbf{M}_0$  which will be used to define the joining path segment.

Let  $C_{ijkl}$  and  $C_{i'j'k'l'}$  be the two constraints that are critical at  $p(0)$ , then  $\mathbf{N}_0$  lies in the plane of the bisector of  $C_{ijkl}$  and  $C_{i'j'k'l'}$ .  $\mathbf{N}_0$  is normalized so that  $\mathbf{N}_0 \cdot \mathbf{N}_{ijkl}(\mathbf{q}_0) = 1$  and it follows that  $\mathbf{N}_0 \cdot \mathbf{N}_{i'j'k'l'}(\mathbf{q}_0) = 1$ . A second vector  $\mathbf{M}_0$  is chosen so that  $\mathbf{M}_0 + \mathbf{N}_{ijkl}(\mathbf{q}_0) = \mathbf{N}_0$ , and so:

$$\mathbf{N}_0 = \frac{\mathbf{N}_{ijkl} + \mathbf{N}_{i'j'k'l'}}{1 + \mathbf{N}_{ijkl} \cdot \mathbf{N}_{i'j'k'l'}} \quad (24)$$

$$\mathbf{M}_0 = \frac{\mathbf{N}_{i'j'k'l'} - (\mathbf{N}_{ijkl} \cdot \mathbf{N}_{i'j'k'l'})\mathbf{N}_{ijkl}}{1 + \mathbf{N}_{ijkl} \cdot \mathbf{N}_{i'j'k'l'}} \quad (25)$$

where  $\mathbf{N}_{ijkl}$  (shorthand for  $\mathbf{N}_{ijkl}(\mathbf{q}_0)$ ) is the normal vector to the critical constraint  $C_{ijkl}$  at orientation  $\mathbf{q}_0$ , and  $\mathbf{N}_{i'j'k'l'} = \mathbf{N}_{i'j'k'l'}(\mathbf{q}_0)$  is the vector normal to the critical constraint  $C_{i'j'k'l'}$ .

We can now define the joining homotopy  $J_2 : I \times I \rightarrow \mathfrak{R}^3 \times SO(3)$  which deforms the path segment  $p'|_{[0, \frac{1}{2}]}$  (with  $p'$  reparametrized so that its domain is  $I$ ):

$$J_2(t, u) = \begin{cases} p(0) + 2t\mathbf{N}_0, & \text{if } t \in [0, \frac{1}{2}u]; \\ p(0) + u\mathbf{N}_0, & \text{if } t \in [\frac{1}{2}u, 1 - \frac{1}{2}u]; \\ p(0) + (2 - 2t)\mathbf{M}_0 + u\mathbf{N}_{ijkl}(\mathbf{q}_0), & \text{if } t \in [1 - \frac{1}{2}u, 1]; \end{cases} \quad (26)$$

Now  $\mathbf{N}_0$  lies in the plane of the bisector of the two constraints that are critical at  $p(0)$ , but it is possible that as  $u$  increases,  $J_2(\frac{1}{2}, u)$  leaves the convex part of the diagram before reaching the concave part. That is, there may be bends in the convex part of the diagram which must be tracked. We must therefore stop the deformation at this point by defining

$$u_v = \sup\{u \mid C_{ijkl} \text{ is critical throughout } J_2(\frac{1}{2}, [0, u])\} \quad (27)$$

and once again we define a homotopy which stops points when they reach the diagram:

$$J_3(t, u) = J_2(t, \min(u, u_v, u_c(t))) \quad (28)$$

$$\text{where } u_c(t) = \inf\{u \mid J_2(t, u) \in \text{conc}(V)\}.$$

and then we have:

**Lemma 4.3** Let  $p : I \rightarrow \mathfrak{R}^3 \times SO(3)$  be a path having  $p(I) = p(0) \in \text{conv}(V)$ . Then  $J_3$  is a homotopy of  $p$  and a path  $p'$  such that

- (i)  $p'(0) = p(0)$
- (ii)  $p'([0, 1 - \frac{1}{2}u_v]) \subset \text{conv}(V)$
- (iii)  $p'((1 - \frac{1}{2}u_v, 1]) \cap V \subset \text{conc}(V)$

**Proof.**

Parts (i) and (ii) follow immediately from the definition of  $J_2$  and  $J_3$ . Part (iii) states that points in the interval  $(1 - \frac{1}{2}u_v, 1]$  that are mapped into the diagram by  $p'$  are mapped into the concave part of the diagram. For this we notice that

$$\frac{\partial J_2(t, u)}{\partial u} = N_{ijkl}(\mathbf{q}_0) \quad (29)$$

for  $t > 1 - \frac{1}{2}u$ , while  $J_2(t, u) \in \text{conv}(V)$  if  $t \leq u$ . Therefore for  $t > 1 - \frac{1}{2}u$  the following is true:

$$\frac{\partial C_{i'j'k'l'}(J_2(t_1, u))}{\partial u} \leq 1 \quad (30)$$

$$\frac{\partial C_{ijkl}(J_2(t_1, u))}{\partial u} = 1$$

So as  $u$  increases, all constraints increase no faster than  $C_{ijkl}$ . So another constraint can only become critical if its lowest common ancestor with  $C_{ijkl}$  is a min node, and such a configuration must be in  $\text{conc}(V)$ .

For continuity of  $J_3$ , first we notice that  $J_2$  is continuous, and  $J_3$  will be continuous if  $u_c(t)$  defined in (28) is continuous. For this we notice that the use of  $u_v$  in the min function in (28) guarantees that  $C_{ijkl}$  is critical at configuration  $J_3(t, u)$  for all  $t$  and  $u$ . The rest of the proof of continuity is identical to the proof of continuity of  $u_c(t)$  in lemma 4.1, using the rate of change condition in (30).  $\square$

**Lemma 4.4** If  $p : I \rightarrow F$  is a path having  $p([0, \frac{1}{2}]) = \{p(0)\} \subset \text{conv}(V)$  then  $p$  is homotopic to a path  $p' : I \rightarrow V$  such that  $p'(0) = p(0)$ .

**Proof.** The proof is inductive. We define a sequence of partial homotopies, that is, maps  $J^n : I \times [u^{n-1}, u^n] \rightarrow F$  such that  $J^n(t, u^n) = J^{n+1}(t, u^n)$ . Then we show that the number  $m$  of partial homotopies required is finite.

**Inductive hypothesis.** The input to our construction is a path  $p^n$ , a value of  $u^n \in I$ , and points  $t_0^n$  and  $t_1^n$  in  $I$  such that

- (i)  $p^n(t) \in \text{conv}(V)$  for  $t \leq t_0^n$
- (ii)  $p^n([t_0^n, t_1^n]) = \{p^n(t_0^n)\} \subset \text{conv}(V)$
- (iii)  $C_{ijkl}$  is critical on  $p^n(I)$ .

From  $C_{ijkl}$  we use lemma 4.1 to define a homotopy  $J_1$  for the path segment  $p^n|_{[t_1^n, 1]}$  (reparametrized to  $I$ ). Similarly, let  $C_{i'j'k'l'}$  be a constraint which is critical at  $p^n(t_0^n)$  and whose lowest common ancestor with  $C_{ijkl}$  is a max node. For these two constraints, we use Lemma 4.3 to define an unravelling homotopy  $J_3$  of the path segment  $p^n|_{[t_0^n, t_1^n]}$  (reparametrized to  $I$ ). This gives us a value of  $u_v$  as in (27), and we define  $u^{n+1} = u^n + u_v$ . Now  $J^{n+1}$  can be defined on the range  $u \in [u^n, u^{n+1}]$  as:

$$J^{n+1} = \begin{cases} p^n(t) & \text{for } t \in [0, t_0^n]; \\ J_3\left(\frac{t-t_0^n}{t_1^n-t_0^n}, u-u^n\right) & \text{for } t \in [t_0^n, t_1^n]; \\ J_1\left(\frac{t-t_1^n}{1-t_1^n}, u-u^n\right) & \text{for } t \in [t_1^n, 1]; \end{cases} \quad (31)$$

then  $J^{n+1}$  is a homotopy because  $J_3$  and  $J_1$  agree on their intersection, and  $J_3(0, u) = p^n(t_0^n)$ . We can define a new path  $p^{n+1}(t) = J^{n+1}(t, u^{n+1})$  and points

$$t_0^{n+1} = t_0^n + \frac{u_v}{2}(t_1^n - t_0^n) \quad \text{and} \quad t_1^{n+1} = t_1^n - \frac{u_v}{2}(t_1^n - t_0^n) \quad (32)$$

and it is readily verified that these satisfy the inductive hypothesis.

For the base case, we set  $p^0 = p$ ,  $u^0 = 0$  and  $t_0^0 = 0$ ,  $t_1^0 = \frac{1}{2}$ , which clearly satisfies the inductive hypothesis.

**Finiteness.** For termination, we must show that after a finite number of steps  $m$ ,  $p^m(t_0^m) \in \text{conc}(V)$ . Suppose  $p^n(t_0^n) \notin \text{conc}(V)$ , and let  $C_{i'j'k'l'}$  be the constraint (along with  $C_{ijkl}$ ) which is critical at  $p^n([t_0^{n-1}, t_0^n])$ , i.e. these are the constraints used to define the homotopy  $J_3$  for  $J^n$ . A third constraint  $C_{i''j''k''l''}$  is also critical at  $p^n(t_0^n)$ , by (27). This constraint has a max node lowest common ancestor with  $C_{i'j'k'l'}$ , which implies that

$$\frac{\partial}{\partial u} C_{i'j'k'l'}(J_3(\frac{1}{2}, u)) < \frac{\partial}{\partial u} C_{i''j''k''l''}(J_3(\frac{1}{2}, u)) \quad (33)$$

and from (24) it follows that

$$(N_{ijkl} + N_{i'j'k'l'}) \cdot N_{i'j'k'l'} < (N_{ijkl} + N_{i'j'k'l'}) \cdot N_{i''j''k''l''} \quad (34)$$

which implies

$$N_{ijkl} \cdot N_{i'j'k'l'} < N_{ijkl} \cdot N_{i''j''k''l''} \quad (35)$$

but the condition (35) defines a total ordering on the constraints distinct from  $C_{ijkl}$ . That is, as  $u$  increases and  $p^n(t_0^n)$  is deformed according to some  $J_2$ , a new constraint  $C_{i''j''k''l''}$  can only become critical if it satisfies condition (35). Once  $C_{i''j''k''l''}$  has become critical,  $C_{i'j'k'l'}$  can never again become critical. Thus we need define homotopies  $J^n$  at most once for each constraint, and so their number  $m$  is bounded by the number of constraints.

We then construct  $J_1$  for the path segment  $p^m|_{[t_1^m, 1]}$  reparameterized to  $I$ . The final homotopy is defined for the range  $u \in [u^m, 1]$  as:

$$J^{m+1}(t, u) = \begin{cases} p^m(t) & \text{for } t \in [0, t_1^m]; \\ J_1\left(\frac{t-t_1^m}{1-t_1^m}, u-u^m\right) & \text{for } t \in [t_1^m, 1]; \end{cases} \quad (36)$$

then the homotopy

$$J_4(t, u) = J^n(t, u) \quad \text{with } u \in [u^{n-1}, u^n] \quad \text{for } n = 1, \dots, m+1 \quad (37)$$

is continuous, and defines a homotopy between  $p(t) = J_4(t, 0)$  and a path  $p'(t) = J_4(t, 1)$  such that  $p'(I) \subset V$ .  $\square$

**Theorem 4.1** Let  $p : I \rightarrow F$  be a path with endpoints in  $V$ . Then  $p$  is path homotopic in  $F$  to a path  $p'$  with the same endpoints which lies entirely in  $V$ .

**Proof.** We first apply the homotopy of lemma 4.2 to all path segments  $p|_{[t_n, t_{n+1}]}$  with an endpoint in  $\text{conv}(V)$ . This does not displace endpoints in  $V$ . Then we construct a global homotopy  $J$  by pasting together homotopies  $J_1$  for path segments with an endpoint in  $\text{conc}(V)$ , and homotopies  $J_4$  for the remaining path segments. The resulting homotopy is continuous if these homotopies agree at their endpoints. Firstly, both  $J_1$  and  $J_4$  do not displace endpoints in  $V$ . Therefore they agree at endpoints in  $V$ . At free endpoints both satisfy the *free endpoint condition*: Assume that after reparametrization,  $p(1)$  is a free endpoint. Then

$$J_i(1, u) = p(1) + \min(u, u_c) \mathbf{N}_{ijkl} \quad \text{for } i = 1, 4 \quad (38)$$

$$\text{with } u_c = \inf\{u \mid p(1) + u \mathbf{N}_{ijkl} \in \text{conc}(V)\}$$

thus  $J$  is continuous, and we define  $p'(t) = J(t, 1)$ . Since  $J_1(I, 1) \subset V$  and  $J_4(I, 1) \subset V$ , we have  $p'(I) \subset V$ .  $\square$

Finally, suppose that in a motion planning problem we are given a start configuration  $(\mathbf{x}, \mathbf{q})$  which is not on  $V$ . Then exactly one constraint  $C_{ijkl}$  is critical there. We apply the homotopy  $J_1$  to the constant path at  $(\mathbf{x}, \mathbf{q})$  to attain the diagram; that is, we plan a straight-line path in direction  $\mathbf{N}_{ijkl}(\mathbf{q})$  to reach  $V$  from the start.

The completeness condition for motion planning has the following simple algebraic formulation. Let  $i : V \hookrightarrow F$  be the inclusion map. Then if  $V$  is a euclidean Voronoi diagram, then it is a strong deformation retract of  $F$ , and hence  $i$  induces an isomorphism of fundamental groups. In our case we have the weaker completeness condition that  $i$  induces an epimorphism:

**Corollary.** (*Algebraic formulation of the completeness condition for motion planning*). Let  $i : V \hookrightarrow F$  be the inclusion map of the simplified Voronoi diagram in free space, with  $y_0 \in V$ , and let  $\pi_1(X, x)$  denote the fundamental group of  $X$  with base point  $x$ . Then the induced homomorphism  $i_* : \pi_1(V, y_0) \rightarrow \pi_1(F, y_0)$  is surjective.

Hence the fundamental group of  $F$  is isomorphic to the quotient group  $\pi_1(V, y_0) / \ker i_*$ . This quotient measures the structural difference between  $F$  and  $V$ .

## 5 Complexity bounds

We have given a definition of the simplified Voronoi diagram  $V$  in the configuration space of a polyhedron in 3-space. This definition does not constitute an algorithm, so our bounds depend on the algorithm used with the diagram. We assume that the diagram will be used as input to a version of the roadmap algorithm of Canny (1986a). This algorithm computes one-dimensional skeletons of semi-algebraic sets in time  $(d^{O(r^2)}n^r \log n)$  for a semi algebraic set defined by  $n$  polynomials of degree  $d$  in  $r$  variables. In our case the number of variables and the degree of the equations are constants.

A naive bound on the complexity of computing a skeleton of  $V$  would be  $O(n^{12} \log n)$  if we are given  $n$  constraints, because the diagram is a subset of the zero sets of all  $O(n^2)$  bisectors of constraints. This bound can be reduced to  $O(n^7 \log n)$  by noticing that the diagram has a simple stratification (decomposition into a union of disjoint manifolds). The diagram is a subset of the set of all  $m$ -sectors, where an  $m$ -sector is the set of points where  $m$  constraints have the same value. If the constraints are in general position, each  $m$ -sector is a manifold of codimension  $m - 1$ . There are  $O(n^m)$   $m$ -sectors, and by the codimension condition,  $m$  must be less than or equal to 7. The complexity of computing the skeleton of this stratification is  $O(n^7 \log n)$ .

While its worst case bounds are poor, the actual performance of the algorithm is expected to be much better, because  $V$  approximates the euclidean Voronoi diagram, as shown in figures 1 and 2. The evidence for this is that the complexity of the euclidean Voronoi diagram for a set of  $n$  points in  $r$  dimensions is  $O(n^{\lfloor \frac{r+1}{2} \rfloor})$ , and the euclidean Voronoi diagram for disjoint line segments in the plane has linear size.

This conjecture is supported by some experimental evidence. We have implemented an algorithm for constructing the simplified Voronoi diagram for the following configuration spaces:  $\mathcal{R}^2$ , the case of an arbitrary polygon translating in the plane amidst polygonal obstacles, and  $\mathcal{R}^2 \times S^1$ , which allows the moving polygon to rotate as well as translate. In many cases the size of  $V$  has been observed to remain roughly linear, as in fig. 1, which our implementation produced.

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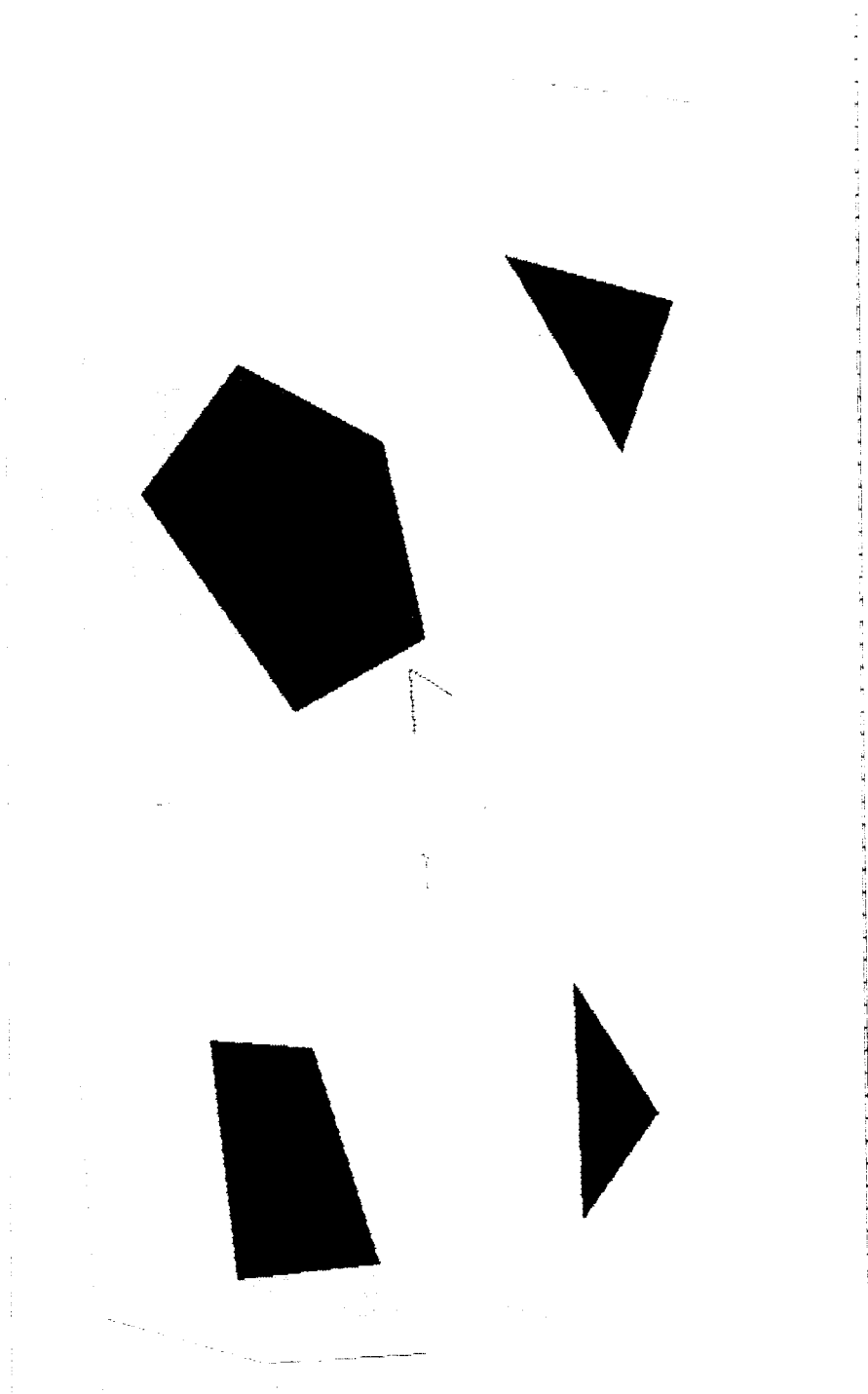


Figure 1. The decomposition of the polygon in the plane. For each position are the triangles  $T_1$  and  $T_2$ . The triangle is the moving object. The shaded polygons are the obstacles in space. The simplified motion diagram is a collection of line segments.



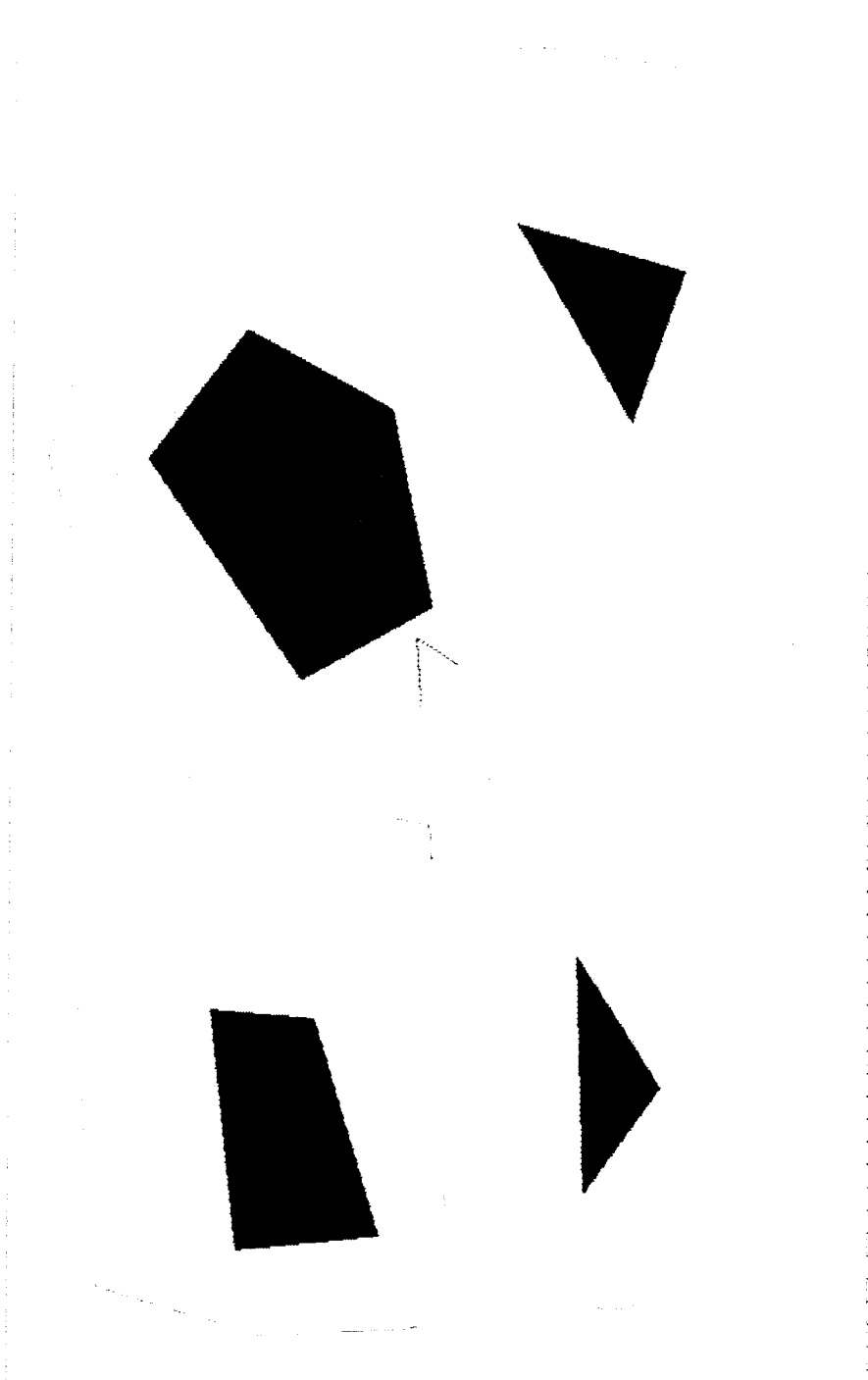


Figure 1. The four types of shapes used in the study.

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