INTEGRAL CONVEX POLYHEDRA

AND

AN APPROACH TO INTEGRALIZATION

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August 1970

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ACKNOWLEDGEMENT

I would like to express my gratitude to:

Professor Chung L. Liu for his inspired teaching, his confidence and warm personal guidance;

Professor Frederick C. Hennie for affording me some of my most valued experiences as a graduate student;

Professor Daniel J. Kleitman for his encouragement;

Professor Jeremy F. Shapiro for his beneficial criticism and suggestions;

Dr. Prakash G. Hebalkar, Professor Suhas S. Patil, Dr. Dean H. Vanderbilt and Mr. Burton J. Smith for their helpful discussions;

My wife, Kay, for her confidence and understanding;

Miss Marsha E. Baker for applying her technical typing skill to this task;

And to the National Science Foundation for their fellowship support.

"Work reported herein was supported (in part) by Project MAC, an M.I.T. research program sponsored by the Advanced Research Projects Agency, Department of Defense, under Office of Naval Research Contract Number Nonr-4102(01). Reproduction in whole or in part is permitted for any purpose of the United States Government."

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Abstract

Many combinatorial optimization problems may be formulated as integer linear programming problems -- that is, problems of the form: given a convex polyhedron P contained in the non-negative orthant of n-dimensional space, find an integer point in P which maximizes (or minimizes) a given linear objective function. Well known linear programming methods would suffice to solve such a problem if:

(i) P is an integral convex polyhedron, or

(ii) P is transformed into the integral convex polyhedron that is the convex hull of the set of integer points in P, a process which is called integralization.

This thesis provides some theoretical results concerning integral convex polyhedra and the process of integralization. Necessary and sufficient conditions for a convex polyhedron P to have the integral property are derived in terms of the system of linear inequalities defining P. A number-theoretic method for integralizing two-dimensional convex polyhedra is developed which makes use of a generalization of the division theorem for integers. The method is applicable to a restricted class of higher dimensional polyhedra as well.

*This report reproduces a thesis of the same title submitted to the Electrical Engineering Department, Massachusetts Institute of Technology, in partial fulfillment of the requirements for the degree of Doctor of Philosophy, August 1970.

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CHAPTER 1

INTRODUCTION

It is well known that many optimization problems of a combinatorial nature may be formulated as integer programming problems. Several authors have written on this subject, including Balinski [1] and [2], Dantzig [5], Hoffman [14], and Hu [17]. Outstanding examples are the problems of constructing a maximal flow in a transport network and constructing a maximal matching in a bipartite graph. In fact, these two examples lead very naturally to integer linear programming formulations which may be solved by standard linear programming methods. Other combinatorial optimization problems, such as the traveling salesman problem or the problem of properly coloring the regions of a map with a minimum number of different colors, lead to integer linear programming formulations which cannot be solved by standard linear programming methods. They require different and more complex solution methods.

In 1958, Gomory [8] developed the first general algorithm for solving integer linear programming problems. There has been considerable interest in integer linear programming since Gomory's pioneering work, and a variety of methods for dealing with the problem have been developed. Balinski [1] and [2] presents an extensive survey of many of these methods and describes some computational experience with them.

In 1965, Gomory [9] brought a new algebraic approach to bear on the problem. He showed that when certain simplifying assumptions are made, the integer linear programming problem can be transformed into an optimization problem involving a finite Abelian group. The algebraic ideas contained in his approach have been further developed and applied by Gomory [10] and [11], Shapiro [23] and [24], Hu [16], and others.

Although a good deal of success in dealing with integer linear programming problems has been achieved, we believe there remains much to be learned in this area. Motivated primarily by the importance of integer linear programming as a tool for solving difficult combinatorial optimization problems, this thesis seeks to provide some relevant theoretical results which may contribute to a better understanding of the problem.

The thesis is organized into seven chapters. In the remainder of this first chapter we review some n-dimensional geometric concepts and definitions related to convex polyhedra. We then give some graphical examples to illustrate the concept of an integral convex polyhedron and the concept of transforming a convex polyhedron into an integral convex polyhedron — a process which we call integralization. We conclude the chapter with a discussion of the relevance of integral convex polyhedra and the process of integralization to the integer linear programming problem, and a description of some of the related research of others.

In Chapter 2 we derive several results concerning convex polyhedra having the integral property. In particular, we present a necessary and sufficient condition for a system of linear equations with integer coefficients and integer right hand sides to have an integer solution. We then use this to extend the results of Hoffman and Kruskal [15] and obtain a necessary and sufficient condition for a particular convex polyhedron to have the integral property.

In Chapter 3 we define a new kind of integral property which we call the totally integral property. This definition is motivated by two considerations. First, the class of totally integral polyhedra is a proper subclass of the class of integral polyhedra. Second, the class of linear programming polyhedra having the totally integral property can be characterized algebraically, by applying some of the group theoretic ideas developed by Gomory [9].

In the next three chapters we turn to the process of integralization. We shall classify polyhedra using two parameters n and k. By an (n,k)-polyhedron we shall mean an n-dimensional polyhedron defined by a system of linear inequalities whose coefficient matrix has rank k, where $k \leq n$.

In Chapter 4 we show that an (n,k)-polyhedron, where n is arbitrary, $n \geq k$, can always be transformed into a (k,k)-polyhedron by a transformation which preserves integralization. This allows us to focus on the problem of integralizing (k,k)-polyhedra without loss of generality. As we shall see, if we know how to integralize a (k,k)-polyhedron then we know how to integralize an (n,k)-polyhedron for arbitrary $n \geq k$. We then discuss the simple problem of transforming and integralizing (n,1)-polyhedra. We show that this is accomplished by using the greatest integer function applied to rational numbers, an application of the division theorem for integers.

In Chapters 5 and 6 we deal with integralization of (2,2)-polyhedra. We introduce a generalization of the division theorem for integers which applies to ordered pairs of integers and use this, together with an associated division process, to integralize such polyhedra.

Chapter 7 concludes the thesis with a discussion of the difficulty we encountered in attempting to generalize the integralization process to make it apply to (k,k)-polyhedra, for $k \geq 3$. We then offer some suggestions for future research in this area.

1.1 Geometric Concepts and Definitions

In this section we review some n-dimensional geometric concepts and definitions. Most of the material is well known, with the exception of the subsection on integral convex sets, and may be found in one or more of the following references: Benson [3], Bonnice and Klee [4], Goldman [6], Goldman and Tucker [7], Grünbaum [12], Hadley [13], Hoffman and Kruskal [15], Klee [19], and Weyl [26].

Let R be the real field. Let R^n be the n-dimensional vector space over R. We refer to $x \in R^n$, $x = (x_1, x_2, \ldots, x_n)$, as a real point or real n-vector (column).

1.1.1 Dimension, Boundedness, and Convexity

Let $x^0, x^1, \ldots, x^k$ be real n-vectors. A real n-vector x is said to be a linear combination of $x^0, x^1, \ldots, x^k$ if

$$x = \lambda_0 x^0 + \lambda_1 x^1 + \cdots + \lambda_k x^k$$

for some real numbers $\lambda_0, \lambda_1, \ldots, \lambda_k$. If the λ's satisfy the constraint,

$$\lambda_0 + \lambda_1 + \cdots + \lambda_k = 1$$

then x is said to be an affine combination of $x^0, x^1, \ldots, x^k$. If the λ's satisfy the additional constraint,

$$\lambda_i \geq 0, \quad \text{for } i = 0, 1, \ldots, k$$

then x is said to be a convex combination of $x^0, x^1, \ldots, x^k$.

A set $X = \{x^0, x^1, \ldots, x^k\}$ is said to be linearly dependent if $\bar{0}$, the all zero n-vector, is a linear combination of $x^0, x^1, \ldots, x^k$ in which some $\lambda_i \neq 0$. If, in addition,

$$\lambda_0 + \lambda_1 + \cdots + \lambda_k = 0$$

then X is said to be affinely dependent. If X fails to be linearly (affinely) dependent then X is said to be linearly (affinely) independent. We shall say that X is a dependent (independent) set iff X is affinely dependent (independent). It can be shown that X is a dependent (independent) set iff the set $\{x^1-x^0, x^2-x^0, \ldots, x^k-x^0\}$ is linearly dependent (independent).

Example 1-1 Independent sets in R^3 are: a single real point, two distinct real points, three real points not contained in a single line, four real points not contained in a single plane. $\square$

For the following definitions we let X be any subset of R^n, either a finite subset or an infinite subset.

X is k-dimensional, where $0 \leq k \leq n$, if X contains an independent set of k+1 real points but none of k+2 real points.

X is bounded if there exist two real n-vectors x^- and x^+ such that, for all $x \in X$, $x^- \leq x \leq x^+$, where $\leq$ is componentwise. Otherwise X is unbounded.

X is convex if, for all $x^1, x^2 \in X$ and for all real λ, $0 \leq \lambda \leq 1$, $\lambda x^1 + (1-\lambda) x^2 \in X$. In words, a convex subset $X \subseteq R^n$ is one which contains the line segment joining any two of its points.

An intersection of convex subsets of R^n is again a convex subset of R^n.

Example 1-2 In R^2, any single real point is a 0-dimensional bounded convex set. R^2 itself is a 2-dimensional unbounded convex set. Fig. 1-1 shows several subsets of R^2 which have various combinations of these properties. □

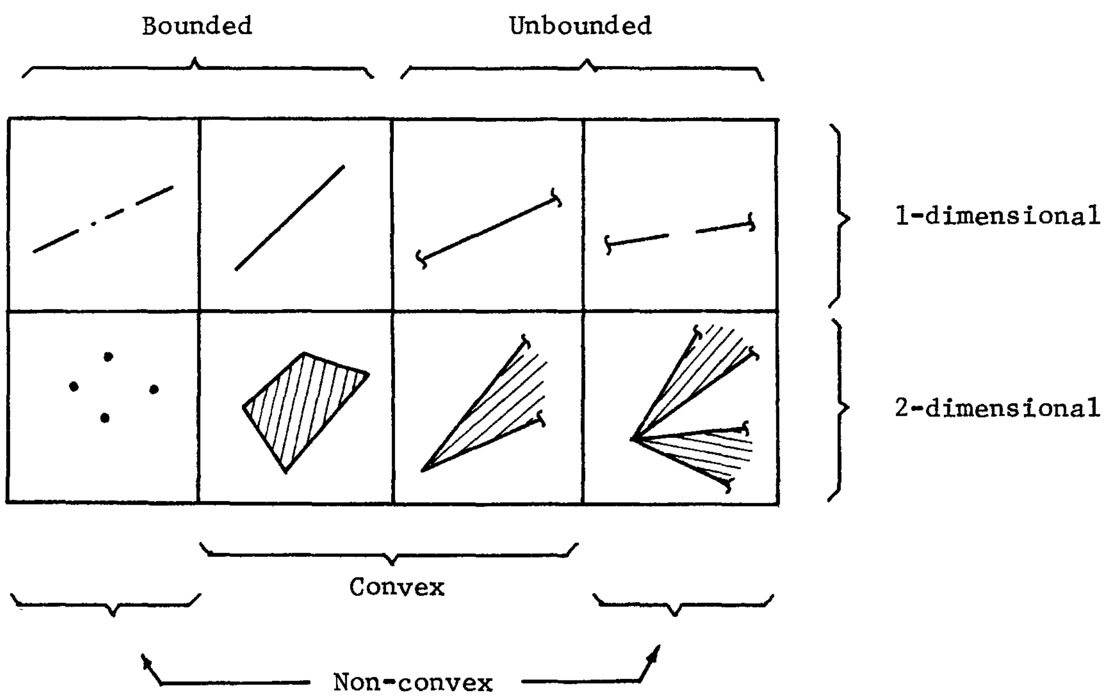


Fig. 1-1

1.1.2 Convex Hull

Given an arbitrary subset $X \subseteq R^n$, which may or may not be convex, it is always possible to find a convex subset of R^n which contains X. R^n itself will do the job. The convex hull or convex closure of X, denoted by H(X), is defined to be the intersection of all convex subsets of R^n which contains X. H(X) is unique. Intuitively, H(X) is the smallest convex subset of R^n which contains X. If X is itself convex, then H(X) = X.

Example 1-3 Fig. 1-2 illustrates several subsets of R^2 and, immediately to the right of each, their convex hulls. □

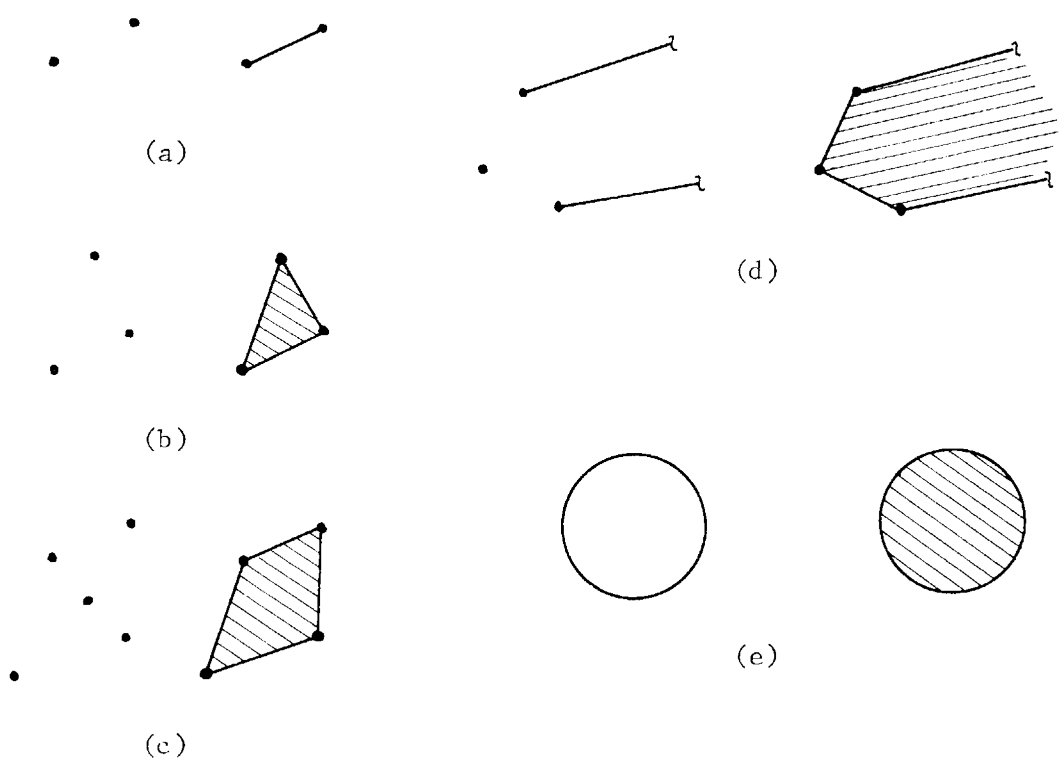


Fig. 1-2

If X is a finite subset of R^n, $X = \{x^0, x^1, \ldots, x^m\}$, then it can be shown (see Benson [3]) that the convex hull of X is given by,

$$H(X) = \left\{ x \in R^n \mid x = \sum_{i=0}^{m} \lambda_i x^i, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\} \tag{1-1}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_m$ are real numbers. In words, (1-1) says that H(X) is the set of all convex combinations of X.

Another characterization of the convex hull of a subset of R^n, similar to (1-1) but more general in that it applies to infinite subsets as well, is given by Caratheodory's Theorem (see Bonnice and Klee [4]). Given a subset $X \subseteq R^n$, the theorem states that

$$H(X) = \left\{ x \in R^n \mid x = \sum_{i=0}^{n} \lambda_i x^i, \lambda_i \geq 0, \sum_{i=0}^{n} \lambda_i = 1, x^i \in X \right\} \tag{1-2}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_n$ are real numbers. In words, (1-2) states that H(X) is the set of all convex combinations of all finite subsets of X containing n+1 or fewer points.

1.1.3 Flats

Let $X = \{x^0, x^1, \ldots, x^k\}$ be an independent set of real points in R^n, where $0 \leq k \leq n$. A k-flat π in R^n is a k-dimensional convex subset of R^n,

$$\pi = \left\{ x \in R^n \mid x = \sum_{i=0}^{k} \lambda_i x^i, \ \sum_{i=0}^{k} \lambda_i = 1 \right\} \tag{1-3}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_k$ are real numbers. π is the set of all affine combinations of X. X is said to generate π. A k-flat in R^n is also known as a k-dimensional affine subspace of R^n.

Equivalently, a k-flat π is a set,

$$\pi = \{x \in R^n \mid A\,x = b\} \tag{1-4}$$

where A is a fixed real $(n-k) \times n$ matrix of rank n-k, and b is a fixed real (n-k)-vector. Both x and b are column vectors, and A x is the matrix product of A and x. (1-3) and (1-4) are equivalent under the assumption that $A\,x^i = b$, for $i = 0, 1, \ldots, k$.

An (n-1)-flat is commonly known as a hyperplane. From (1-4) we see that a k-flat is an intersection of n-k independent hyperplanes. If $k = n$, the condition $A\,x = b$ in (1-4) vanishes. Thus the only n-flat is R^n itself. A 0-flat is just a single real point.

Example 1-4 In R^3, 0,1,2,3-flats are respectively: a single real point, a line, a plane, and R^3 itself.

□

A 0-flat is a bounded set. A k-flat, where $1 \leq k \leq n$, is an unbounded set.

Given $k + 1$ independent real points in R^n, there is one and only one k-flat containing these real points, and none of smaller dimension.

1.1.4 Half-flats

A half k-flat τ in R^n, where $1 \leq k \leq n$, is a k-dimensional unbounded convex subset of R^n,

$$\tau = \{x \in R^n \mid A\,x = b \quad \text{and} \quad \alpha\, x \leq \beta\} \tag{1-5}$$

where $A\,x = b$ defines a k-flat, α is a fixed real n-vector (row) which is not in the row space of A, and β is a fixed real number. The set

$$\{x \in R^n \mid A\,x = b \quad \text{and} \quad \alpha\, x = \beta\}$$

is a (k-1)-flat known as the boundary (k-1)-flat of the half k-flat τ.

Example 1-5 In R^3, half 1,2,3-flats are respectively: a half-line, a half-plane, and a half-space. The corresponding boundary 0,1,2-flats are respectively: a point, a line, and a plane. □

A half n-flat,

$$\{x \in R^n \mid \alpha\, x \leq \beta\} \tag{1-6}$$

where α is a fixed real n-vector (row), $\alpha \neq 0$, and β is a fixed real number, is commonly known as a half-space. Its boundary (n-1)-flat is a boundary hyperplane. An intersection of a half-space (1-6) and a half-space,

$$\{\underline{x} \in R^n \mid \alpha\, x \geq \beta\} \tag{1-7}$$

is the boundary hyperplane common to both (1-6) and (1-7).

Let X be a subset of R^n. A half-space in R^n which contains X, and whose boundary hyperplane contains one or more real points in X, is known as a supporting half-space for X. Its boundary hyperplane is known as a supporting hyperplane for X.

1.1.5 Convex Polyhedra

A convex polyhedron P in R^n is a non-empty intersection of a finite number of half-spaces in R^n,

$$P = \{x \in R^n \mid A\, x \leq b\}$$

where A is a fixed real $m \times n$ matrix and b is a fixed real m-vector (column). We shall say that P is an (n,k)-polyhedron if the rank of A is k.

P may be bounded or unbounded, and may have any dimension between 0 and n inclusive. Flats and half-flats are special cases of convex polyhedra.

The structure of a convex polyhedron is best understood by considering its faces. A <u>face</u> of a convex polyhedron P is a non-empty intersection of P with one or more supporting hyperplanes.

Let $a\,x \leq b$ be any one inequality in the system $A\,x \leq b$ defining P. The half-space

$$\tau = \{x \in R^n \mid a\,x \leq b\}$$

may or may not be a supporting half-space for P. τ is a supporting half-space if

$$\{x \in P \mid a\,x = b\} \tag{1-8}$$

is non-empty. In this case, (1-8) is a face of P.

Let S be an arbitrary subset of $\{1, 2, \ldots, m\}$. Let A_S denote the submatrix of A comprised of those rows of A whose indices appear in S, and let b_S denote the subvector of b comprised of those components of b whose indices appear in S. A subset P_S of P,

$$P_S = \{x \in P \mid A_S\,x = b_S\}$$

if non-empty, is a face of P. Furthermore, every face of P may be obtained in this manner.

If P_S is a face of P, and we let S', where $S' \subseteq S$, identity a maximal set of linearly independent rows of A_S, then $P_{S'} = P_S$. Thus we may generate all faces P_S of P by considering only those sets of indices S which identify linearly independent sets of rows of A. We refer to such sets of indices as <u>face subsets</u>. We include the case $S = \varphi$, where φ is the empty set, so that P, which is equal to P_{φ}, is a face of itself. Since there are only a finite number of ways to form S, P has a finite number of faces. The faces of P are themselves convex polyhedra.

If a face of P has dimension k, we refer to that face as a <u>k-face</u> of P. 0, 1-faces of P, if they exist, are commonly known as <u>vertices</u> and <u>edges</u> of P, respectively.

If P_S and $P_{S'}$ are faces of P and $P_S \subseteq P_{S'}$, then P_S is a <u>subface</u> of $P_{S'}$. Clearly, if $S' \subset S$ then P_S is a subface of $P_{S'}$.

A <u>minimal face</u> of P is one which has no proper subface.

Let P be a convex (n,k)-polyhedron, where $0 \leq k \leq n$. Hoffman and Kruskal [15] show that a face P_S of P is minimal iff A_S consists of k linearly independent rows of A. Furthermore, they show that if P_S is a minimal face of P, then

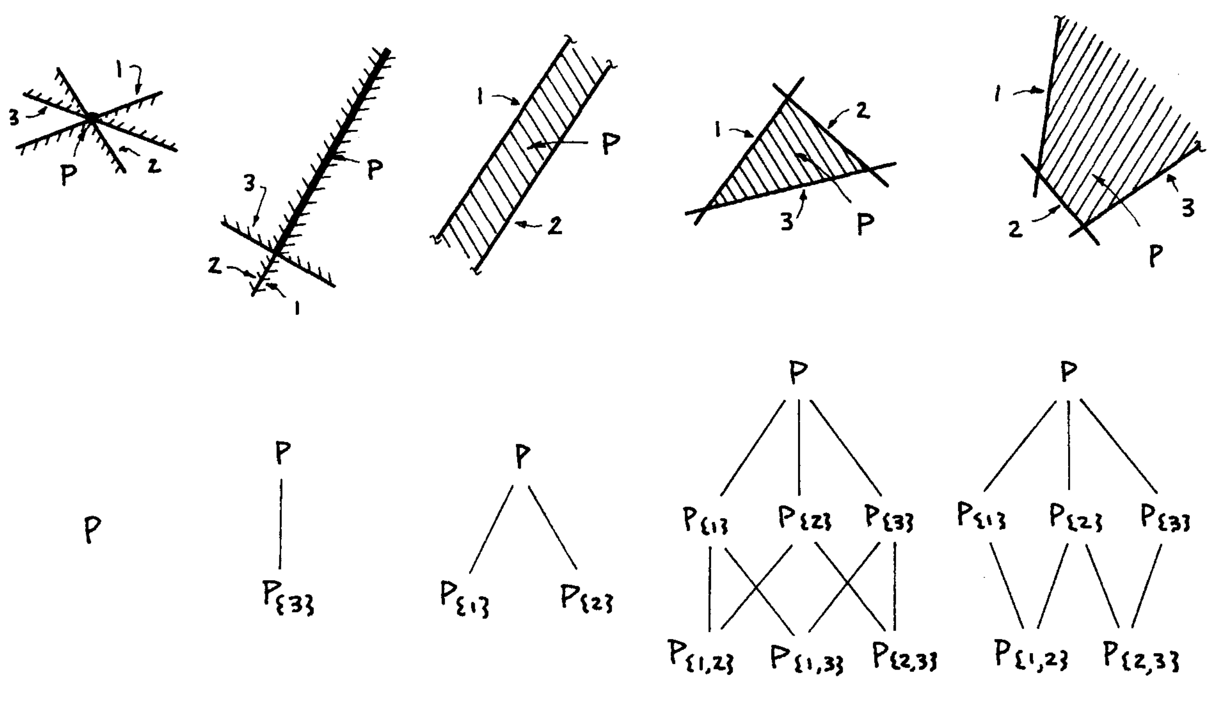
$$P_S = \{x \in R^n \mid A_S x = b_S\}$$

That is, all minimal faces of P are (n-k)-flats.

The set of all faces of a convex (n,k)-polyhedron P is partially ordered by $\subseteq$. The greatest member of this partially ordered set is P itself. There is no least member, in general. All faces fall into ranks, each rank being composed of all faces of a given dimension. The minimal faces of P, which are all (n-k)-flats, constitute the lowest rank. The next rank consists of all (n-k+1)-faces of P, and so on. The highest rank consists of P alone, whose dimension lies between n-k and n inclusive.

Example 1-6 Fig. 1-3 shows graphical representations of several convex polyhedra in R^2 and partial ordering diagrams displaying the face structure of each. □

The boundary of a face of a convex polyhedron P is the union of all proper subfaces of that face. A boundary point of a face is a real point in the boundary of that face. A minimal face of P has no boundary; that is, its boundary is empty. The boundary of P itself is the union of all proper faces of P.



(a) (b) (c) (d) (e)

Fig. 1-3

If P is a bounded convex polyhedron, then every face of P is bounded. Since every minimal face of P is flat, and the only bounded flat is a 0-flat, it follows that every minimal face of P is a 0-flat (or vertex), and that P is an (n,n)-polyhedron.

A bounded convex polyhedron P is equal to the convex hull of its vertices (see Hadley [10]). Thus the vertices of P completely specify P.

There is a similar result which applies to any convex polyhedron, bounded or unbounded. In order to state it, we first generalize the notion of a minimal face.

A face of a convex polyhedron P may or may not be equal to the convex hull of its boundary. A face of P is said to be <u>reducible</u> if it is equal to the convex hull of its boundary, and <u>irreducible</u> if it is not. Flats and half-flats are irreducible faces.

Klee [19] shows that a face of a convex polyhedron P is irreducible iff that face is a flat or a half-flat. He shows further that P is equal to the convex hull of its irreducible faces.

A minimal face of P is irreducible, but an irreducible face of P is not necessarily minimal. The converse does not hold because a half-flat is irreducible but not minimal. The notions of an irreducible face and a minimal face coincide in the case of bounded convex polyhedra since no face of a bounded convex polyhedron can be an (unbounded) half-flat.

If an irreducible face of P is a half-flat, then its boundary is a flat which is also an irreducible face of P. A <u>maximal irreducible face</u>

of P is one which is not contained in another irreducible face of P. Klee [19] proves the following sharper result:

A convex polyhedron P is equal to the convex hull of its maximal irreducible faces.

Example 1-7 The minimal, irreducible, and maximal irreducible faces of the convex polyhedra represented in Fig. 1-3 are tabulated below:

Fig. 1-3	Bounded/ Unbounded	Minimal Faces	Irreducible Faces	Maximal Irreducible Faces
(a)	Bounded	P	P	P
(b)	Unbounded	$P_{\{3\}}$	P, $P_{\{3\}}$	P
(c)	Unbounded	$P_{\{1\}}$, $P_{\{2\}}$	$P_{\{1\}}$, $P_{\{2\}}$	$P_{\{1\}}$, $P_{\{2\}}$
(d)	Bounded	$P_{\{1,2\}}$, $P_{\{1,3\}}$, $P_{\{2,3\}}$	$P_{\{1,2\}}$, $P_{\{1,3\}}$, $P_{\{2,3\}}$	$P_{\{1,2\}}$, $P_{\{1,3\}}$, $P_{\{2,3\}}$
(e)	Unbounded	$P_{\{1,2\}}$, $P_{\{2,3\}}$	$P_{\{1\}}$, $P_{\{3\}}$, $P_{\{1,2\}}$, $P_{\{2,3\}}$	$P_{\{1\}}$, $P_{\{3\}}$

$\square$

A class of convex polyhedra of special interest are those arising in connection with linear programming problems. A linear programming problem has the form:

$$\text{Maximize} \quad c\,x$$

$$\text{Subject to} \quad A\,x \leq b \tag{1-9}$$

$$x \geq 0 \tag{1-10}$$

where c is a fixed real n-vector (row), A is a fixed real $m \times n$ matrix, and b is a fixed real m-vector. The linear function cx is known as an objective function. A linear programming problem may instead involve minimizing an objective function. By simply negating c we may change such a minimization problem into a maximization problem. The set P of real points satisfying the inequalities (1-9) and (1-10),

$$P = \{x \in R^n \mid A\,x \leq b \quad \text{and } x \geq 0\} \tag{1-11}$$

is a convex polyhedron. We shall refer to P as a linear programming polyhedron. The condition $x \geq 0$ gives P some special characteristics which we now examine.

Suppose the inequalities (1-9) and (1-10) are consolidated and (1-11) is rewritten as

$$P = \{x \in R^n \mid A'\,x \leq b'\}$$

where A' contains as a submatrix the $n \times n$ identity matrix (negated). The rank of A' is n, and it follows that P is an (n,n)-polyhedron, all of whose minimal faces are 0-flats.

What about the irreducible faces of P? It can be shown that $\{x \in R^n \mid x \geq 0\}$, known as the non-negative orthant of R^n, contains no 1-flat. It follows that the non-negative orthant of R^n contains no k-flat, for $1 \leq k \leq n$, and no half k-flat, for $2 \leq k \leq n$. Since P is wholly contained in the non-negative orthant of R^n, P contains no k-flat, for $1 < k \leq n$, and no half k-flat for $2 \leq k \leq n$. Thus the irreducible faces of P, necessarily being flats or half-flats, must be 0-flats or half 1-flats.

P is equal to the convex hull of a finite number of 0-flats and/or half 1-flats.

1.1.6 Integral Convex Sets

Let J be the ring of integers. Let J^n be the n-dimensional module over J. We refer to $x \in J^n$ as an integer point or an integer n-vector.

For any subset $X \subseteq R^n$, we define I(X) to be the set of integer points contained in X,

$$I(X) = \{x \in J^n \mid x \in X\}.$$

A convex subset $X \subseteq R^n$ is said to be <u>integral</u> (or to have the <u>integral property</u>) iff X is the convex hull of the set of integer points contained in X, that is, iff $X = H(I(X))$. Since $I(X) \subseteq X$ it follows that $H(I(X)) \subseteq H(X) = X$. Thus X is integral iff $X \subseteq H(I(X))$, that is, iff every $x \in X$ is a convex combination of integer points in X.

We are primarily interested in convex polyhedra having the integral property or, more simply, integral convex polyhedra. In Chapter 2 we develop necessary and sufficient conditions for a convex polyhedron to have the integral property.

As the following example shows, not every integral convex set is an integral convex polyhedron.

<u>Example 1-8</u> Let $X = \{x^0, x^1, x^2, \ldots\}$ be the infinite subset of J^2 defined inductively as follows:

$$x^0 = (0, 0)$$

$$x^{n+1} = x^n + (n+1, 1).$$

Fig. 1-4 shows the first few members of X and a portion of the boundary of H(X). H(X) is an integral convex set because every $x \in H(X)$ is a convex combination of integer points in H(X). Clearly H(X) is equal to the intersection of an infinite number of half-planes and therefore is not a convex polyhedron.

□

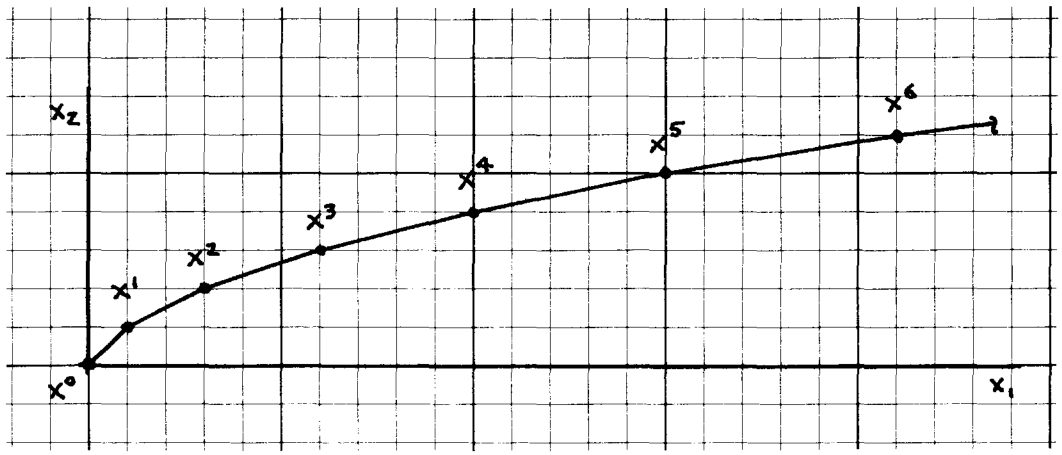


Fig. 1-4

If X is a convex subset of R^n then there exists a unique convex subset X' of R^n defined by X' = H(I(X)). In words, X' is the convex hull of the set of integer points contained in X. We point out two extreme cases. First, if X is an integral convex set then X' = H(I(X)) = X. Second, if X contains no integer points then X' = H(φ) = φ, where φ is the empty set.

The following lemma is a direct consequence of our definitions.

Lemma 1-1

Let X and X' be convex subsets of R^n. Then X' = H(I(X)) iff

(i) X' is integral

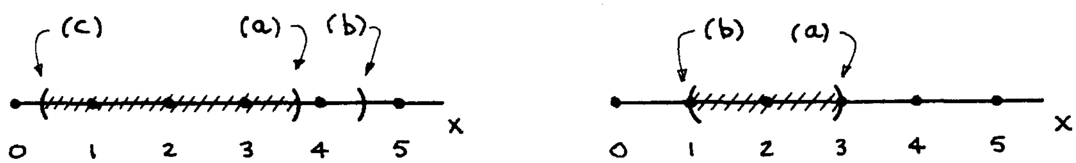
and

(ii) I(X') = I(X).

Example 1-9   Consider the following system of linear inequalities:

$$\begin{aligned} &(a) \quad 4x \le 15 \\ &(b) \quad 2x \le 9 \\ &(c) \quad -3x \le -1 \end{aligned} \qquad (1\text{-}12)$$

The shaded interval of Fig. 1-5(i) represents the set of real points x which satisfy (1-12).



(i)   (ii)

Fig. 1-5

Some of these real points are integer points and these are indicated with dots. In Fig. 1-5(ii) the shaded interval is the convex hull of these integer points. It represents the set of real points which satisfy the following system of linear inequalities:

$$(a) \quad x \leq \left\lfloor \frac{15}{4} \right\rfloor = 3$$

$$(b) \quad -x \leq \left\lfloor \frac{-1}{3} \right\rfloor = -1$$

where $\lfloor \lambda \rfloor$ denotes the greatest integer less than or equal to the real number λ. The shaded interval of Fig. 1-5(i) is an example of a convex polyhedron. The shaded interval of Fig. 1-5(ii) is an example of an integral convex polyhedron. The process of transforming the former into the latter is an example of integralization.

□

Example 1-10 Consider the following system of linear inequalities:

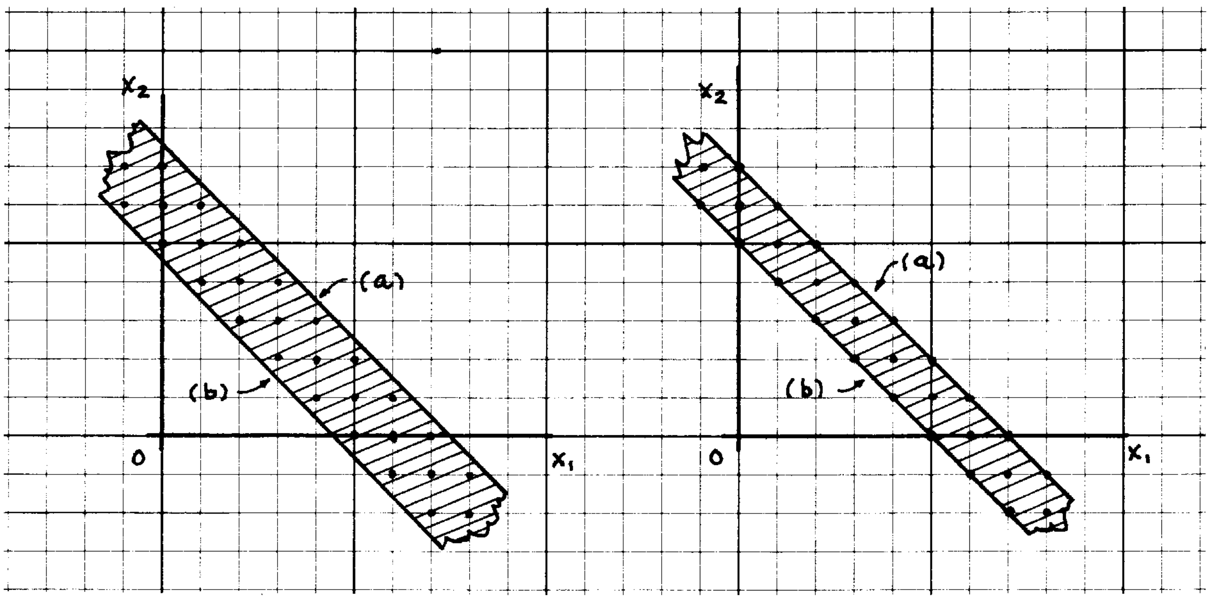
$$\begin{aligned} &(a) \quad 2x_1 + 2x_2 \leq 15 \\ &(b) \quad -4x_1 - 4x_2 \leq -18 \end{aligned} \tag{1-13}$$

The shaded region of Fig. 1-6(i) represents the set of real points $x = (x_1, x_2)$ which satisfy (1-13). It contains a countably infinite number of integer points, some of which are indicated with dots. In Fig. 1-6(ii) the shaded region is the convex hull of these integer points. It represents the set of real points which satisfy the following system of linear inequalities:

$$(a) \quad x_1 + x_2 \leq \left\lfloor \frac{15}{2} \right\rfloor = 7$$

$$(b) \quad -x_1 - x_2 \leq \left\lfloor \frac{-18}{4} \right\rfloor = -5$$

In Fig. 1-6(i) we have an example of an unbounded convex polyhedron, and in Fig. 1-6(ii) an example of an unbounded integral convex polyhedron. Again, the transformation of the former into the latter is an example of integralization. □



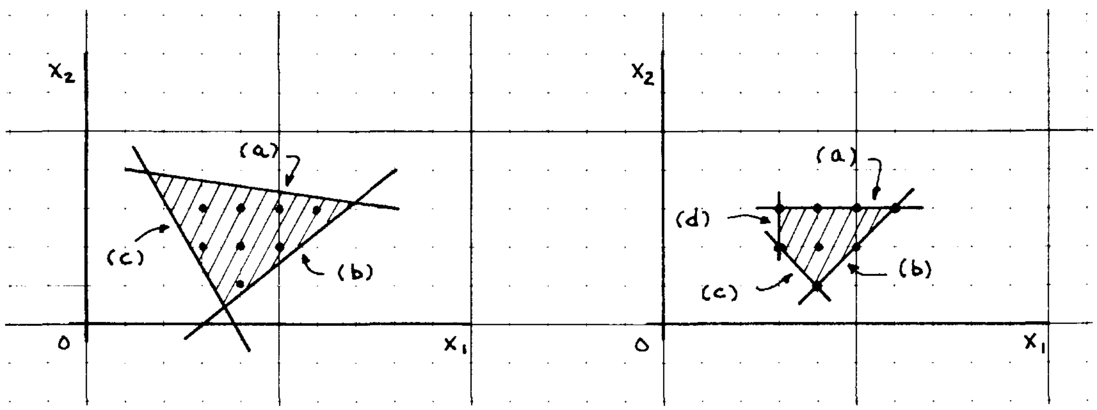
(i) (ii)

Fig. 1-6

Example 1-11 Consider the system

$$\begin{aligned}
&(a) \quad x_1 + 7x_2 \leq 29 \\
&(b) \quad 4x_1 - 5x_2 \leq 12 \qquad\qquad (1\text{-}14) \\
&(c) \quad -7x_1 - 4x_2 \leq -27
\end{aligned}$$

Fig. 1-7(i) illustrates the convex polyhedron defined by (1-14), while Fig. 1-17(ii) illustrates the integral convex polyhedron which is the convex hull of integer points contained in the former.



(i) (ii)

Fig. 1-7

The integral convex polyhedron of Fig. 1-7(ii) is defined by the system of inequalities,

$$\begin{array}{llcr}
\text{(a)} & \qquad x_2 & \leq & 3 \\
\text{(b)} & x_1 - x_2 & \leq & 3 \\
\text{(c)} & -x_1 - x_2 & \leq & -5 \\
\text{(d)} & -x_1 & \leq & -3.
\end{array}$$

If we were to imagine the x_1-x_2 plane as a peg-board, with pegs placed in all integer point positions, then a rubber band in tension, made to conform with the boundary of the shaded region of Fig. 1-7(i), would, when released, assume the form of the boundary of the shaded region of Fig. 1-7(ii), assuming that the rubber band remains in tension at the completion of its contraction. In its final state, the rubber band is supported entirely by pegs. Every vertex of the region it encloses is an integer point. This is a physical method for integralizing the bounded convex polyhedron of this example.

In three dimensional space, a rubber sphere made to conform with the boundary of a bounded convex polyhedron and enclosing small marbles somehow fixed in integer point positions would do the job. In higher dimensional space it is difficult to imagine any physical process for accomplishing integralization. What is nedded, of course, is a mathematical procedure for performing the integralization transformation. □

In the chapters to come we shall be primarily concerned with two questions:

(i) Given a convex polyhedron, how does one tell if it is an integral polyhedron?

(ii) Given a non-integral convex polyhedron, how does one integralize it?

We answer the first question in Chapter 2 and we answer the second question for a restricted class of polyhedra in Chapters 4, 5 and 6.

1.3 Integer Linear Programming and Related Research

In this section we seek to accomplish two goals. Our first goal is to establish the relevance of integral convex polyhedra and the process of integralization to integer linear programming. Our second goal is to describe some of the research done by others concerning integral convex polyhedra and integralization.

1.3.1 Integer Linear Programming

As we stated in the preceding section, a linear programming problem has the following form:

$$\left.\begin{array}{lll} \text{Maximize} & c\,x & \\ \text{Subject to} & A\,x & \leq b \\ & x & \geq 0 \end{array}\right\} \qquad (1\text{-}15)$$

where c is a fixed real n-vector, A is a fixed real $m \times n$ matrix, and b is a fixed real m-vector. The inequalities in (1-15) define a convex polyhedron P.

A commonly used algorithm for solving (1-15) is the simplex method developed by Dantzig [5]. The fundamental idea of the method is: if an optimal solution to (1-15) exists then some vertex of P is an optimal solution. Beginning with any vertex of P, the simplex method generates a sequence of vertices of P for which the objective function cx takes on non-decreasing values. The algorithm terminates when an optimum vertex is found, or gives an indication that no optimal solution exists.

The simplex method has proven to be an efficient algorithm and has found many applications in operations research, management science and economics.

An integer linear programming problem has the form:

$$\left.\begin{array}{ll} \text{Maximize} & c\,x \\ \text{Subject to} & A\,x \le b \\ & \quad x \ge 0 \\ & x \text{ an integer vector} \end{array}\right\} \qquad (1\text{-}16)$$

where c is a fixed integer n-vector, A is a fixed $m \times n$ matrix, and b is a fixed integer m-vector. There are variations to (1-16), notably the more general formulation in which all constants are allowed to be real. In another variation, only a fixed subset of the components of x are required to be integers; the remaining components are allowed to be real. We shall take (1-16) to be the standard integer linear programming problem.

Again, the inequality constraints of (1-16) define a convex polyhedron P. The simplex method, when applied to (1-16), does not work in general simply because it is no longer true that if an optimal solution to (1-16) exists then some vertex of P is an optimal solution. Some or all of the vertices of P may not be integer points. Said in another way, optimal solutions to (1-16), if they exist, may be embedded in the interior of P and not accessible to any vertex generating algorithm.

If P happens to be an integral convex polyhedron then, as we shall see in Chapter 2, every vertex of P is an integer point and the simplex method will produce an optimal solution to (1-16) for any objective function cx.

Suppose P is not an integral convex polyhedron, as is usually the case. If we integralize P to obtain the integral convex polyhedron P', where $P' = H(I(P))$, then the simplex method applied to P' will produce an optimum solution to (1-16) for any objective function cx. We acknowledge the fact that there is a certain amount of naivety in this approach to integer linear programming. This stems from the fact that P' is potentially a very complex object — complex in terms of the number of inequalities needed to define P' —

and it certainly seems inefficient to construct P' in order to discover just one of its vertices.

However, we offer the following observation, made by Gomory, in support of the idea of integralization as an important subject of research. We quote from Gomory [11]:

"Since any algorithm for the integer programming problem, whether related to linear programming, branch and bound, exhaustive search, or whatever, must end up finding a vertex of P', information on P' seems relevant to any approach to the integer programming problem. Yet information about P' is very difficult to obtain."

Although the last sentence of this quotation does not contribute to our argument, we include it to avoid mis-representation by omission.

It has been our goal to gain some insight into the structural relationship between P' and P by focusing on the integral property and the process of integralization for convex polyhedra in general, not just linear programming polyhedra. We believe that such insight can contribute to the improvement of existing integer linear programming algorithms and the development of new ones.

1.3.2 Related Research

In this subsection we describe some of the known results related to integral convex polyhedra and integralization.

The major results concerning integral convex polyhedra known to us are contained in the paper by Hoffman and Kruskal [15]. They consider convex polyhedra defined by system of linear inequalities $Ax \leq b$ in which all constants are integers, and define such a convex polyhedron P to be integral iff every minimal face of P contains an integer point. If the minimal faces of P are vertices then this definition says that P is integral iff every vertex of P is an integer point. As we shall see in Chapter 2, their definition is equivalent to our own.

Hoffman and Kruskal obtain the following result concerning the integral property for convex polyhedra. Let A be an integer $m \times n$ matrix of rank k. Let S denote a subset of row indices of A and let A_S denote the submatrix comprised of rows of A identified by S. Let gcd $[A_S]$ denote the greatest common divisor of all $|S|$-order minors of A_S. Then $Ax \leq b$ defines an integral convex polyhedron P(b) for every integer m-vector b iff

$$\gcd\ [A_S] = 1 \tag{1-17}$$

for all k-subsets S of linearly independent rows of A. We observe that for a particular integer m-vector b, condition (1-17) is sufficient for P(b) to be integral but not necessary.

Hoffman and Kruskal also obtain a result concerning the integral property for linear programming polyhedra. An integer $m \times n$ matrix is said to have the unimodular property iff every minor (of every order) of A has value 1, 0, or -1. Entries in a unimodular matrix are therefore restricted to be 1, 0, or -1. They show that the inequalities $Ax \leq b$ and $x \geq 0$ define an integral convex polyhedron Q(b) for every integer m-vector b iff A is a unimodular matrix. We observe that for a particular integer m-vector b, the condition that A be unimodular is sufficient for Q(b) to be integral but not necessary. Unimodular matrices occur in the integer linear programming formulations of several combinatorial optimization problems, including the network flow problem and the maximal matching problem for bipartite graphs.

Concerning integralization, Gomory in [10] are [11] gives a method for integralizing a corner polyhedron P — that is, a polyhedron defined by $Ax \leq b$ where A is an integer $n \times n$ matrix of rank n and b is an integer n-vector. Basically the method involves generating a large but finite number k of auxiliary inequalities,

$$c^i x \geq 1, \quad i = 1, 2, \ldots, k$$

where c^i, for $i = 1, 2, \ldots, k$, is obtained as an integer solution to an equation over a finite Abelian group derived from the matrix A. Gomory shows that these inequalities define a polyhedron whose vertices correspond one-to-one to inequalities defining P', where $P' = H(I(P))$, and indicates how to obtain the latter from the former. Thus the problem of integralizing a corner polyhedron becomes one of finding all vertices of an auxiliary polyhedron, which can be accomplished by linear programming methods.

We mentioned previously that the end product P' of integralizing a convex polyhedron P is potentially a very complex object. Rubin [22] obtains an interesting quantitative result concerning this complexity. Let $a_1, a_2, a_3, \ldots, a_k, \ldots$ be the infinite sequence of integers defined by the recurrence,

$$a_k = a_{k-1} + a_{k-2} \qquad \text{for} \quad k \geq 3$$

$$a_1 = 1, \; a_2 = 1.$$

This is the well known Fibonacci sequence. Rubin defines an infinite sequence $P_1, P_2, P_3, \ldots, P_k, \ldots$ of bounded (2,2)-polyhedra based on the Fibonacci sequence as follows. P_k, for $k \geq 1$, is defined by the three inequalities,

$$a_{2k}\, x_1 + a_{2k+1}\, x_2 \leq a_{2k+1}^2 - 1$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

P_k, for $k \geq 1$, has 3 vertices and 3 edges. He shows that P_k', where $P_k' = H(I(P_k))$, for $k \geq 1$, has exactly k+3 vertices and k+3 edges. Thus for a 2-dimensional bounded polyhedron P, it may require arbitrarily many inequalities to define $P' = H(I(P))$, even though P is defined by just three inequalities.

CHAPTER 2

THE INTEGRAL PROPERTY IN CONVEX POLYHEDRA

In this chapter we develop several results concerning convex polyhedra having the integral property. Using our definition of an integral convex set X as one which is the convex hull of the set I(X) of integer points in X, we derive necessary and sufficient conditions for convex polyhedra to have the integral property.

The chapter is divided into three sections. In the first we deal with integral flats, in the second with integral half-flats, and in the third with integral convex polyhedra in general.

2.1 Integral Flats

A k-flat π in R^n is generated by taking all affine combinations of an independent set of k+1 real points. Lemma 2-1 tells us when π is integral in terms of its generating set.

Lemma 2-1

A k-flat π is integral iff π is generated by an independent set $\{x^0, x^1, \ldots, x^k\}$ of integer points.

<u>Proof</u>

We first prove necessity of the condition. Assume π is an integral k-flat. Since π is a k-flat, π is generated by an independent set Y of k+1 real points. Since π is integral, each real point in Y is a convex combination of integer points in $I(\pi)$. The dimension of $I(\pi)$ is at most k. If $I(\pi)$ has dimension less than k, then Y has dimension less than k, which is a contradiction. Thus $I(\pi)$ has dimension k and π contains an independent set of k+1 integer points which generates π.

In order to prove sufficiency of the condition, assume π is generated by an independent set $\{x^0, x^1, \ldots, x^k\}$ of integer points. We shall show that every $x \in \pi$ is a convex combination of integer points in π. It will then follow that π is integral.

Let x be an arbitrary real point in π,

$$x = \lambda_0 x^0 + \lambda_1 x^1 + \cdots + \lambda_k x^k$$

where $\lambda_0, \lambda_1, \ldots, \lambda_k$ are real numbers and

$$\lambda_0 + \lambda_1 + \cdots + \lambda_k = 1.$$

Equivalently we may write

$$x = x^0 + \lambda_1(x^1-x^0) + \lambda_2(x^2-x^0) + \cdots + \lambda_k(x^k-x^0). \qquad (2\text{-}1)$$

Let functions $h_0\colon R \to J$ and $h_1\colon R \to J$ be defined as follows. Let $h_0(\lambda_i)$ be the greatest integer less than or equal to λ_i and let $h_1(\lambda_i)$ be the smallest integer greater than λ_i. (Thus $h_1(\lambda_i) - h_0(\lambda_i) = 1$, for all real λ_i.) let $\{0,1\}^k$ be the set of all binary k-tuples, a representative member of which is denoted by $b = (b_1, b_2, \ldots, b_k)$. We claim that x is a convex combination of the 2^k integer points,

$$x^b = x^0 + \sum_{i=1}^{k} h_{b_i}(\lambda_i)\ (x^i - x^0), \qquad \text{all } b \in \{0,1\}^k$$

in π. (Geometrically, these 2^k integer points are vertices of a k-dimensional cube in π and we claim that x lies in this cube.)

In order to exhibit this convex combination we define functions g_0: $R \to R$ and g_1: $R \to R$ as follows:

$$g_0(\lambda_i) = h_1(\lambda_i) - \lambda_i$$

$$g_1(\lambda_i) = \lambda_i - h_0(\lambda_i).$$

We then define function g_b: $R^k \to R$ as follows:

$$g_b(\lambda) = \prod_{i=1}^{k} g_{b_i}(\lambda_i).$$

We claim that

$$x = \sum_{b \in \{0,1\}^k} g_b(\lambda)\ x^b \tag{2-2}$$

where

$$0 \leq g_b(\lambda) \leq 1 \qquad \text{for all } b \in \{0,1\}^k \tag{2-3}$$

and

$$\sum_{b \in \{0,1\}^k} g_b(\lambda) = 1. \tag{2-4}$$

In (2-2), $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is the real k-tuple of coefficients appearing in (2-1).

In order to prove this claim we first observe that (2-3) holds since $0 < g_0(\lambda_i) \leq 1$ and $0 \leq g_1(\lambda_i) < 1$ for all real numbers λ_i.

Next we prove (2-4) by induction on k. For $k = 1$ we have

$$g_0(\lambda_1) + g_1(\lambda_1) = h_1(\lambda_1) - h_0(\lambda_1) = 1.$$

Assume (2-4) is true for some k, $1 \leq k \leq n$. Then

$$\sum_{b \in \{0,1\}^{k+1}} \left[\prod_{i=1}^{k+1} g_{b_i}(\lambda_i) \right]$$

$$= \left[g_0(\lambda_{k+1}) + g_1(\lambda_{k+1}) \right] \cdot \sum_{b \in \{0,1\}^k} \left[\prod_{i=1}^{k} g_{b_i}(\lambda_i) \right]$$

$$= g_0(\lambda_{k+1}) + g_1(\lambda_{k+1})$$

$$= h_1(\lambda_{k+1}) - h_0(\lambda_{k+1}) = 1.$$

Finally we prove that (2-2) is valid. We have

$$\sum_{b \in \{0,1\}^k} g_b(\lambda)\ x^b$$

$$= \sum_{b \in \{0,1\}^k} g_b(\lambda) \left[x^0 + \sum_{i=1}^{k} h_{b_i}(\lambda_i)\ (x^i - x^0) \right]$$

$$= \left[\sum_{b \in \{0,1\}^k} g_b(\lambda) \right] x^0 + \sum_{b \in \{0,1\}^k} g_b(\lambda) \left[\sum_{i=1}^{k} h_{b_i}(\lambda_i)(x^i - x^0) \right]$$

$$= x^0 + \sum_{i=1}^{k} \left[\sum_{b \in \{0,1\}^k} g_b(\lambda)\ h_{b_i}(\lambda_i) \right] (x^i - x^0).$$

The expression within brackets may be expanded as

$$\sum_{b \in \{0,1\}^k} g_b(\lambda)\ h_{b_i}(\lambda_i)$$

$$= \sum_{b \in \{0,1\}^k} \left[\prod_{j=1}^{k} g_{b_j}(\lambda_j) \right] h_{b_i}(\lambda_i)$$

$$= \left[g_0(\lambda_i)\, h_0(\lambda_i) + g_1(\lambda_i)\, h_1(\lambda_i) \right] \cdot \frac{1}{2} \sum_{b \in \{0,1\}^k} \left[\prod_{\substack{j=1 \\ j \neq i}}^{k} g_{b_j}(\lambda_j) \right]$$

$$= g_0(\lambda_i)\, h_0(\lambda_i) + g_1(\lambda_i)\, h_1(\lambda_i)$$

$$= (h_1(\lambda_i) - \lambda_i)\, h_0(\lambda_i) + (\lambda_i - h_0(\lambda_i))\, h_1(\lambda_i)$$

$$= \lambda_i (h_1(\lambda_i) - h_0(\lambda_i))$$

$$= \lambda_i$$

Thus

$$\sum_{b \in \{0,1\}^k} g_b(\lambda)\, x^b = x^0 + \sum_{i=1}^{k} \lambda_i (x^i - x^0)$$

$$= x$$

and we have expressed an arbitrary $x \in \pi$ as a convex combination of integer points in π. It follows that π is integral. □

Let A be an integer $(n-k) \times m$ matrix of rank n-k and let b be an integer n-vector. Our next objective is to determine what conditions must be imposed on A and b in order for $Ax = b$ to define an integral k-flat. The preceding lemma allows us to rephrase the question as: What conditions must be imposed on A and b in order for the solution space of $Ax = b$ to contain an independent set of k+1 integer points.

We first introduce the following notation. If B is an integer $r \times s$ matrix of rank r, where $1 \leq r \leq s$, then we define gcd [B] to be the greatest common divisor of all $\binom{s}{r}$ r^{th}-order minors of B, where $\binom{s}{r}$ is a binomial coefficient. Notice that if $r = s$ then B is a square matrix and gcd [B] = |det B|, while if $r = 1$ then B is a row vector and gcd [B] is the greatest common divisor of all components of B.

Theorem 2-1

Let A be an integer $(n-k) \times n$ matrix of rank n-k and let b be an integer (n-k)-vector. Then

$$A\,x = b$$

defines an integral k-flat π iff

$$\gcd\,[A] = \gcd\,[A \mid b].$$

Proof

Assume that $A\,x = b$ defines an integral k-flat π. By Lemma 2-1, π is generated by an independent set of k+1 integer points. Let x^0 be an integer point in π. Then $A\,x^0 = b$. Let S denote an arbitrary (n-k-1)-subset of column indices of A and let A^S be the $(n-k) \times (n-k-1)$ submatrix of A comprised of those columns of A identified by S. Since

$$\det\,[A^S \,\vdots\, b] = \det\,[A^S \,\vdots\, Ax^0]$$

it follows that $\det\,[A^S \vdots b]$ is an integer combination of $(n-k)^{th}$-order minors of A. Thus

$$\gcd\,[A] \;\Big|\; \det\,[A^S \,\vdots\, b].$$

Since this holds for any S, we conclude that

$$\gcd\,[A] = \gcd\,[A \,\vdots\, b].$$

Now we assume that

$$\gcd\,[A] = \gcd\,[A \,\vdots\, b]$$

and prove that $Ax = b$ defines an integral k-flat π.

According to Jacobson [18], there exists an integer $(n-k) \times (n-k)$ matrix R with $\det R = \pm 1$ and an integer $n \times n$ matrix C with $\det C = \pm 1$ such that

$$A = R\,\bar{A}\,C$$

where $\bar{A}$ is an integer $(n-k) \times n$ diagonal matrix,

$$\bar{A} = \left[\begin{array}{cccc|c} \bar{a}_1 & & & & \\ & \bar{a}_2 & & & \\ & & \ddots & & 0 \\ & & & \bar{a}_{n-k} & \end{array}\right].$$

Furthermore, Jacobson [18] proves that

$$\gcd\,[A] = \gcd\,[\bar{A}].$$

It is easily seen that

$$\gcd\,[\bar{A}] = \bar{a}_1\ \bar{a}_2\ \cdots\ \bar{a}_{n-k}.$$

We may write $A\,x = b$ as

$$R\ \bar{A}\ C\ x = b$$

or as

$$\bar{A}\ y = \bar{b}$$

where

$$y = C\ x$$
$$\bar{b} = R^{-1}\ b.$$

If we let C' be the following integer $(n+1) \times (n+1)$ matrix

$$C' = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & 1 \end{array}\right]$$

we see that $\det C' = \det C = \pm 1$ and

$$[A \mid b] = R\ [\bar{A} \mid \bar{b}]\ C'.$$

Again, it follows from Jacobson [18] that

$$\gcd\,[A \mid b] = \gcd\,[\bar{A} \mid \bar{b}].$$

We now have

$$\begin{array}{ccc} \gcd\,[A] & = & \gcd\,[A \mid b] \\ \parallel & & \parallel \\ \gcd\,[\bar{A}] & & \gcd\,[\bar{A} \mid \bar{b}] \end{array}$$

from which we conclude that

$$\gcd \left[\begin{array}{cccc|c} \bar{a}_1 & & & 0 & \\ & \bar{a}_2 & & & \\ & & \ddots & & 0 \\ 0 & & & \bar{a}_{n-k} & \end{array}\right] = \gcd \left[\begin{array}{cccc|c|c} \bar{a}_1 & & & 0 & & \bar{b}_1 \\ & \bar{a}_2 & & & & \bar{b}_2 \\ & & \ddots & & 0 & \vdots \\ 0 & & & \bar{a}_{n-k} & & \bar{b}_{n-k} \end{array}\right].$$

Thus

$$\begin{gathered} \bar{a}_1\, \bar{a}_2 \cdots \bar{a}_{n-k} \mid \bar{b}_1\, \bar{a}_2 \cdots \bar{a}_{n-k} \\ \bar{a}_1\, \bar{a}_2 \cdots \bar{a}_{n-k} \mid \bar{a}_1\, \bar{b}_2 \cdots \bar{a}_{n-k} \\ \vdots \\ \bar{a}_1\, \bar{a}_2 \cdots \bar{a}_{n-k} \mid \bar{a}_1\, \bar{a}_2 \cdots \bar{b}_{n-k} \end{gathered}$$

and

$$\begin{gathered} \bar{a}_1 \mid \bar{b}_1 \\ \bar{a}_2 \mid \bar{b}_2 \\ \vdots \\ \bar{a}_{n-k} \mid \bar{b}_{n-k}\ . \end{gathered}$$

The k-flat $\bar{\pi}$ defined by $\bar{A}\, y = \bar{b}$ contains the following independent set of k+1 integer points,

$$\begin{bmatrix} \bar{b}_1/\bar{a}_1 \\ \bar{b}_2/\bar{a}_2 \\ \vdots \\ \bar{b}_{n-k}/\bar{a}_{n-k} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \bar{b}_1/\bar{a}_1 \\ \bar{b}_2/\bar{a}_2 \\ \vdots \\ \bar{b}_{n-k}/\bar{a}_{n-k} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \bar{b}_1/\bar{a}_1 \\ \bar{b}_2/\bar{a}_2 \\ \vdots \\ \bar{b}_{n-k}/\bar{a}_{n-k} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} \bar{b}_1/\bar{a}_1 \\ \bar{b}_2/\bar{a}_2 \\ \vdots \\ \bar{b}_{n-k}/\bar{a}_{n-k} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Therefore $\bar{\pi}$ is generated by these k+1 integer points and, by Lemma 2-1, $\bar{\pi}$ is integral. Since the k-flat π defined by $A\, x = b$ is the image of $\bar{\pi}$ under the integer transformation C^{-1}, π is also an integral k-flat, which is what we set out to prove. $\square$

<u>Corollary 2-1</u>

$A\, x = b$ defines an integral k-flat iff $A\, x = b$ has an integer solution.

Proof

The necessity of the condition follows from Lemma 2-1. To prove sufficiency, assume A x = b has an integer solution x^0. Then, as in the necessity part of the proof of Theorem 2-1,

$$\gcd\ [A] = \gcd\ [A \mid b].$$

Thus, by Theorem 2-1, A x = b defines an integral k-flat. $\square$

Suppose that we have a system A x = b of n-k linear equations which defines an integral k-flat. The following corollary shows that any (n-k')-subset, $k \leq k' < n$, of these linear equations defines an integral k'-flat. Let S be an arbitrary (n-k')-subset of $\{1,2,\ldots, n-k\}$, where $k \leq k' < n$. Let A_S denote the submatrix of A comprised of those rows of A whose indices appear in S and let b_S denote the subvector of b comprised of those components of b whose indices appear in S.

Corollary 2-2

If A x = b defines an integral k-flat then $A_S\ x = b_S$ defines an integral k'-flat.

Proof

Suppose A x = b defines an integral k-flat. By Corollary 2-1, A x = b has an integer solution x^0. But x^0 is also an integer solution to $A_S\ x = b_S$. Thus, again by Corollary 2-1, $A_S\ x = b_S$ defines an integral k'-flat. □

2.2 Integral Half-flats

We now consider the conditions under which a half-flat is integral. Let τ be a half k-flat in R^n,

$$\tau = \{x \in R^n \mid A\ x = b \quad \text{and} \quad \alpha\ x \leq \beta\}$$

and let π be its boundary (k-1)-flat. Again, we consider all constants to be integers.

Theorem 2-2

A half k-flat τ is integral iff its boundary (k-1)-flat π is integral.

Proof

Assume τ is integral. Since τ is not empty, $I(\tau)$ is not empty. If π contains an integer point then, by Corollary 2-1, π is integral. Otherwise

$$\alpha\ x < \beta \qquad \text{for all } x \in I(\tau)$$

or

$$\alpha\, x \leq \beta-1 \qquad \text{for all } x \in I(\tau).$$

Thus $A\,x = b$ and $\alpha\, x \leq \beta-1$ define a new half k-flat τ' such that $\tau' \subset \tau$ and $I(\tau') = I(\tau)$. But this contradicts our assumption that τ is integral. Thus π contains an integer point and π is integral.

Now assume that π is integral. We shall argue that τ is the convex hull of a countably infinite number of integral (k-1)-flats, and thus τ is itself integral.

Let

$$\Delta = \gcd \begin{bmatrix} A \\ \hline \alpha \end{bmatrix}$$

and let π_j be the (k-1)-flat defined by

$$\begin{bmatrix} A \\ \hline \alpha \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \\ \hline \beta - j\Delta \end{bmatrix} .$$

It can be shown that τ is the convex hull of the set of (k-1)-flats $\{\pi_0, \pi_1, \pi_2, \cdots\}$. We shall show that each (k-1)-flat π_j in the set is integral.

First of all, π_0 is integral by assumption. We then have from Theorem 2-1 that

$$\gcd \left[\begin{array}{c} A \\ \hline \alpha \end{array}\right] = \gcd \left[\begin{array}{c|c} A & b \\ \hline \alpha & \beta \end{array}\right].$$

We next observe that each $(n-k+1)^{th}$-order minor of

$$\left[\begin{array}{c|c} A & b \\ \hline \alpha & \beta - j\Delta \end{array}\right]$$

which includes the right hand column differs from the corresponding minor of

$$\left[\begin{array}{c|c} A & b \\ \hline \alpha & \beta \end{array}\right]$$

by a multiple of Δ. It follows that

$$\gcd \left[\begin{array}{c} A \\ \hline \alpha \end{array}\right] = \gcd \left[\begin{array}{c|c} A & b \\ \hline \alpha & \beta - j\Delta \end{array}\right]$$

and by Theorem 2-1, π_j is integral. Then, since each $x \in \tau$ is a convex combination of real points in $\pi_0, \pi_1, \pi_2, \ldots$, each of which is a convex combination of integer points in τ, it follows that each $x \in \tau$ is a convex combination of integer points in τ. Thus τ is integral. $\square$

2.3 Integral Convex Polyhedra

We are now prepared to derive some conditions under which a convex polyhedra has the integral property. Let P be a convex polyhedron in R^n,

$$P = \{x \in R^n \mid A\,x \leq b\}.$$

As before, we assume all elements of A and b to be integers.

Theorem 2-3

P is integral iff every face of P is integral.

Proof

Suppose every face of P is integral. Since P is the convex hull of its irreducible faces all of which are integral, P is itself integral.

Conversely, assume P is integral. If P is a flat then its only face is P itself which is integral by assumption. Otherwise P has one or more proper faces. Let P_S be any proper face of P,

$$P_S = \{x \in P \mid A_S x = b_S\}$$

where S specifies a subset of the inequalities defining P.
Let $x \in P_S$. Since $x \in P$, x is a convex combination of integer points $x^1, x^2, \ldots, x^m \in P$,

$$x = \sum_{i=1}^{m} c_i x^i, \quad 0 < c_i < 1, \sum_{i=1}^{m} c_i = 1.$$

We claim that $x^1, x^2, \ldots, x^m \in P_S$, that is

$$A_S x^i = b_S, \quad \text{for} \quad i = 1,2,\ldots, m.$$

Otherwise

$$A_S x^i < b_S \quad \text{for some } i$$

and

$$A_S x = A_S \sum_{i=1}^{m} c_i x^i = \sum_{i=1}^{m} c_i A_S x^i < b_S$$

which contradicts the fact that $x \in P_S$. Thus x is a convex combination of integer points in P_S, and P_S is integral. $\square$

The following corollary gives a necessary and sufficient condition on the system of inequalities $A\ x \leq b$ defining P for P to be integral.

Corollary 2-3

P is integral iff

$$\gcd\ [A_S] = \gcd\ [A_S \mid b_S]$$

for all face subsets S.

Proof

Assume P is integral. By Theorem 2-3 every face P_S of P is integral. Let π_S be the flat defined by $A_S\ x = b_S$. Since $P_S \subseteq \pi_S$, π_S must contain an integer point, and by Corollary 2-1 and Theorem 2-1,

$$\gcd\ [A_S] = \gcd\ [A_S \mid b_S].$$

Now assume that

$$\gcd\ [A_S] = \gcd\ [A_S \mid b_S]$$

for all face subsets S of P. Consider the irreducible faces of P. An irreducible face P_S of P is either a flat or a half-flat. If P_S is a flat then P_S is integral by Theorem 2-1. If P_S is a half-flat then its boundary flat is an irreducible face of P which is integral by Theorem 2-1. Then, by Theorem 2-2, P_S is integral. Since P is the convex hull of its irreducible faces, all of which are integral, P is integral. $\square$

We may further restrict the condition of the preceding corollary. This is done in the following.

Corollary 2-4

P is integral iff

$$\gcd\ [A_S] = \gcd\ [A_S \mid b_S]$$

for all minimal-face subsets S.

Proof

Necessity follows directly from Corollary 2-3. In order to prove that the condition is sufficient, assume that

$$\gcd\ [A_S] = \gcd\ [A_S \mid b_S]$$

for all minimal-face subsets S. Consider the irreducible faces of P. An irreducible face P_S of P is either: (i) a flat which is a minimal face of P, or (ii) a half-flat whose boundary flat is a minimal face of P. In the first case, P_S is integral by Theorem 2-1. In the second case, the boundary flat of P_S is integral by Theorem 2-1, and thus P_S

is integral by Theorem 2-2. Since P is the convex hull of its irreducible faces, all of which are integral, P is integral. □

Corollary 2-4 tells us that P is integral if all of its minimal faces are integral. If P is an (n,k)-polyhedron then the rank of A in $A x \leq b$ is k and the minimal faces of P are all (n-k)-flats. If $k = n$ then the minimal faces of P are all vertices. For example, this is the case for all linear programming polyhedra since the non-negative condition $x \geq 0$ insures a rank n coefficient matrix. In such cases, Corollary 2-4 asserts that P is integral iff every vertex of P is an integer point.

CHAPTER 3

THE TOTALLY INTEGRAL PROPERTY FOR CONVEX POLYHEDRA

In this chapter we define a new property which we call the totally integral property. We show that the class of totally integral polyhedra is a proper subclass of the class of integral polyhedra. We then consider the class of linear programming polyhedra having the totally integral property and show that this class can be characterized algebraically using some of the group theoretic ideas developed by Gomory [9].

3.1 Totally Integral Convex Polyhedra

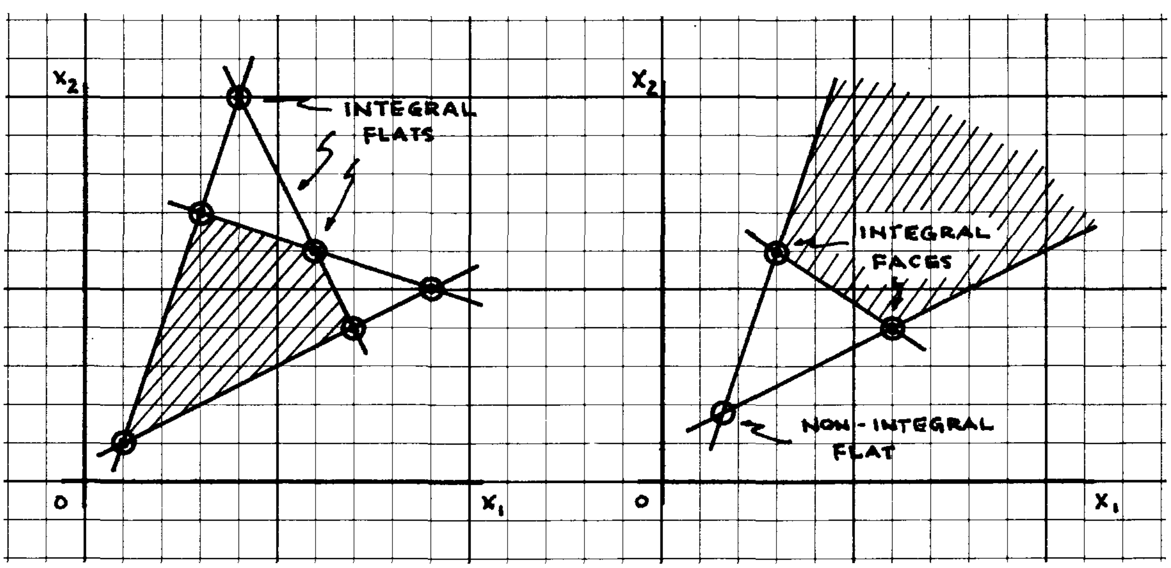
We begin by defining the totally integral property. Let P be a convex polyhedron defined by the system of inequalities $A\,x \leq b$, where, as usual, all elements of A and b are integers. Let S be a subset of row indices identifying a linearly independent set of rows of A. Then the system of equations $A_S\,x = b_S$ defines a flat which we denote by π_S. We shall say that π_S is a *flat of P* and that S is a *flat subset* Notice that a face subset is always a flat subset, but that a flat subset need not be a face subset. It π_S is a flat of P then it is possible that P_S, where

$$P_S = P \cap \pi_S$$

is empty and thus not a face of P.

A convex polyhedron P is said to be totally integral iff every flat π_S of P is integral.

Example 3-1 Fig. 3-1(a) illustrates a totally integral convex polyhedron. Fig. 3-1(b) illustrates an integral convex polyhedron which is not totally integral. □



(a) (b)

Fig. 3-1

The following lemma gives us a necessary and sufficient condition for P to be totally integral, expressed in terms of A and b.

Lemma 3-1

P is totally integral iff

$$\gcd\,[A_S] = \gcd\,[A_S \mid b_S]$$

for all flat subsets S.

Proof

The lemma follows directly from the definition of the totally integral property and Theorem 2-1. □

Suppose matrix A has rank n-k, where $0 \leq k < n$. We shall say that a flat π_S of P is minimal iff S identifies n-k linearly independent rows of A. Thus a minimal flat of P is a k-flat. We may restrict the condition of the preceding lemma as follows.

Lemma 3-2

P is totally integral iff

$$\gcd\,[A_S] = \gcd\,[A_S \mid b_S]$$

for all minimal-flat subsets S.

Proof

Necessity follows directly from Lemma 3-1. To prove sufficiency, assume

$$\gcd\ [A_S] = \gcd\ [A_S \mid b_S]$$

for all minimal-flat subsets S. Let R be any flat subset. We may augment R with additional rows of A, if necessary, to obtain a minimal-flat subset S such that $R \subseteq S$. By assumption,

$$\gcd\ [A_S] = \gcd\ [A_S \mid b_S]$$

and by Corollary 2-2 and Theorem 2-1,

$$\gcd\ [A_R] = \gcd\ [A_R \mid b_R].$$

It follows from Lemma 3-1 that P is totally integral. □

We see immediately that a totally integral convex polyhedron P is always integral. Every minimal face of P is a minimal flat of P. If P is totally integral then every minimal face of P is integral and, by Corollary 2-4, P is integral.

The converse is not true — an integral convex polyhedron P need not be totally integral. It is possible that every minimal face of P is integral, but that one or more minimal flats of P which are not faces do not have the integral property.

Suppose we have an integer $m \times n$ matrix A and we let T(A) be the set of integer m-vectors b such that $A x \leq b$ defines a totally integral convex polyhderon. We show that T(A) forms a group under vector addition.

We first observe that T(A) always contains the all zero m-vector. Applying Lemma 3-1 with b = 0, since

$$\gcd\ [A_S] = \gcd\ [A_S \,\vdots\, 0]$$

for all flat subsets S, it follows that $0 \in T(A)$. Furthermore, T(A) is closed under vector addition. For if b^1 and b^2 are two integer m-vectors in T(A) then

$$\gcd\ [A_S] = \gcd\ [A_S \,\vdots\, b_S^1]$$

and

$$\gcd\ [A_S] = \gcd\ [A_S \,\vdots\, b_S^2]$$

for all flat subsets S, and it follows that

$$\gcd\ [A_S] = \gcd\ [A_S \,\vdots\, b_S^1 + b_S^2]$$

for all flat subsets S. Thus $b^1 + b^2 \in T(A)$. Moreover, T(A) contains inverses for each of its members. If $b \in T(A)$ then

$$\gcd\ [A_S] = \gcd\ [A_S \,\vdots\, b_S]$$

and

$$\gcd\ [A_S] = \gcd\ [A_S \,\vdots\, -b_S]$$

for all flat subsets S. Thus $-b \in T(A)$.

In the following section we completely characterize the group T(A) for linear programming polyhedra.

3.2 Totally Integral Linear Programming Polyhedra

We now focus our attention on those convex polyhedra contained in the non-negative orthant of R^n — that is, convex polyhedra P defined by

$$\begin{aligned} A\,x &\le b \\ x &\ge 0 \end{aligned} \tag{3-1}$$

where again A is an integer $m \times n$ matrix and b is an integer m-vector. Given matrix A, let T(A) denote the set of integer m-vectors b for which (3-1) defines a totally integral convex polyhedron. In the following, we obtain a complete characterization of T(A), using group theoretic methods originally developed by Gomory [9]. Specifically, we show that T(A) may be characterized implicitly as the set of integer solutions to a single equation expressed over a finite Abelian group.

Writing the constraints (3-1) defining P as

$$\begin{bmatrix} A \\ \hdashline -I \end{bmatrix} x \le \begin{bmatrix} b \\ \hdashline 0 \end{bmatrix}$$

where I is an $n \times n$ identity matrix, we see that the coefficient matrix $\begin{bmatrix} A \\ \hdashline -I \end{bmatrix}$ has rank n. It follows that all minimal flats of P are 0-flats (or points), as are all minimal faces of P. From Lemma 3-2, P is totally integral iff all minimal flats of P are integer points. Also, from Corollary 2-4, P is integral iff all minimal faces of P are integer points.

We reformulate the constraints (3-1) defining P in the following equivalent form,

$$[A \mid I] \begin{bmatrix} x \\ \hdashline z \end{bmatrix} = b \tag{3-2}$$

$$x \ge 0, \quad z \ge 0 \tag{3-3}$$

where I is an $m \times m$ identity matrix and z is an m-vector of slack variables. A basic solution to (3-2) is one which is described as follows. Let B be any basis of $[A \mid I]$, i.e. a non-singular $m \times m$ submatrix of $[A \mid I]$, and let x_B be the corresponding m-subvector of $\begin{bmatrix} x \\ \hdashline z \end{bmatrix}$. Let N be the remaining $m \times n$ submatrix of $[A \mid I]$ and let x_N be the corresponding n-subvector of $\begin{bmatrix} x \\ \hdashline z \end{bmatrix}$. We may write (3-2) as,

$$[B \vdots N] \begin{bmatrix} x_B \\ \hdashline x_N \end{bmatrix} = b. \tag{3-4}$$

The unique solution $\begin{bmatrix} x_B \\ \hdashline x_N \end{bmatrix}$ to (3-4) given by

$$x_B = B^{-1} b$$

$$x_N = 0$$

is a basic solution to (3-2). If $x_B \geq 0$ then the basic solution $\begin{bmatrix} x_B \\ \hdashline x_N \end{bmatrix}$ is a basic feasible solution to (3-2), and B is a feasible basis for (3-2). Notice that whether or not a basis B of $[A \vdots I]$ is feasible depends upon the particular right hand side vector b in (3-2).

It is easily seen that 0-flats of P correspond to basic solutions of (3-2) and that 0-faces (or vertices) of P correspond to basic feasible solutions to (3-2). Thus, P is totally integral iff every basic solution to (3-2) is an integer point, and P is integral iff every basic feasible solution to (3-2) is an integer point.

We now take a slightly different point of view in expressing the condition for P to be totally integral. Let B be a basis and let M(B) denote the set of all integer combinations of columns of B. Then the basic solution $\begin{bmatrix} x_B \\ \hdashline 0 \end{bmatrix}$ is an integer point iff the solution x_B to $Bx_B = b$ is an integer point or, equivalently, iff $b \in M(B)$. There are at most $\binom{n+m}{m}$ bases of $[A \mid I]$, which we denote by $B_1, B_2, \ldots, B_k$. It follows that P is totally integral iff

$$b \in M(B_i) \qquad \text{for } i = 1,2, \ldots, k$$

or, equivalently, iff

$$b \in \bigcap_{i=1}^{k} M(B_i).$$

Recalling our defining of T(A) as the set of integer m-vectors b for which P — defined by (3-2) and (3-3) with A fixed — is totally integral, we see that

$$T(A) = \bigcap_{i=1}^{k} M(B_i).$$

We note that $M(B_i)$, for $i = 1,2, \ldots, k$, forms an Abelian group under vector addition. Moreover, this group is a subgroup of the Abelian group M(I) under vector addition, where I is the $m \times m$ identity matrix. It follows that T(A), as an intersection of subgroups of M(I), is also a subgroup of M(I). We summarize these results in the following.

Lemma 3-3

Given a fixed integer $m \times n$ matrix A, the set T(A) of all integer m-vectors b for which the constraints

$$[A \mid I] \begin{bmatrix} x \\ \hline z \end{bmatrix} = b$$

$$x \geq 0, \quad z \geq 0$$

define a totally integral convex polyhedron is a subgroup of M(I). Moreover

$$T(A) = \bigcap_{i=1}^{k} M(B_i)$$

where B_i, for i = 1,2, ..., k, are all the bases of $[A \mid I]$. $\square$

Before proceeding with our characterization of T(A), we should point out that a similar development is not possible for the integral property. The fundamental difficulty with the integral property is its connection with feasible bases. Which bases of $[A \mid I]$ are feasible depends on the particular choice of b as a right hand side vector. Given an integer $m \times n$ matrix A, suppose we let Q(A) be the set of integer m-vectors b for which (3-2) and (3-3) define an integral convex polyhedron. Then, if b^1 and b^2 are in Q(A), it is not true in general that $b^1 + b^2$ is in Q(A). This is due to the fact that the feasible bases associated with b^1 may not be the same as those associated with b^2. Thus Q(A) is not closed under vector addition.

The real advantage of dealing with the totally integral property, and our motivation in defining it, is that it involves all bases of $[A \,\vdots\, I]$ for every choice of right hand side vector b. As a result it leads to an interesting algebraic characterization of T(A).

We begin by considering a single basis B of $[A \,\vdots\, I]$ and showing how M(B) is characterized as the set of integer solutions to an equation expressed over a finite Abelian group whose structure is completely known. An integer m-vector b is in M(B) iff $B^{-1}b$ is an integer m-vector. If we let

$$\Delta = |\det B|$$

then $B^{-1}b$ is an integer m-vector iff

$$\Delta B^{-1}b \equiv 0 \bmod \Delta \tag{3-5}$$

where the columns of ΔB^{-1} are integer m-vectors, 0 is the all zero m-vector, and the congruence applies componentwise. Consider the group $M(\Delta B^{-1})$. If we introduce the congruence relation on $M(\Delta B^{-1})$ which relates two integer m-vectors iff they are congruent modulo Δ componentwise, then the quotient group with respect to this congruence relation is $\dfrac{M(\Delta B^{-1})}{M(\Delta I)}$, where I is the m $\times$ m identity matrix. Moreover, if we let $\beta_1, \beta_2, \ldots, \beta_m$ denote respectively the congruence classes containing each of the columns of ΔB^{-1}, and if we now let 0 denote the congruence class containing the all zero m-vector, then (3-5) may be expressed as the following equation over the group $G = \frac{M(\Delta B^{-1})}{M(\Delta I)}$,

$$\sum_{i=1}^{m} b_i \beta_i = 0. \tag{3-6}$$

Thus an integer m-vector b is in M(B) iff b satisfies the group equation (3-6) over G.

We next show that (3-6) can be replaced by an equation over a new group which is isomorphic to G, whose size and structure is more easily revealed. First we note that if H is any group isomorphic to G, with isomorphism φ: $G \to H$, then the equation (3-6) over G is satisfied by some integer m-vector b iff the following equation over H

$$\sum_{i=1}^{m} b_i \varphi(\beta_i) = \varphi(0)$$

is satisfied by b.

We claim that the group $G = \frac{M(\Delta B^{-1})}{M(\Delta I)}$ is isomorphic to the group $G' = \frac{M(I)}{M(B)}$. The non-singular linear transformation $\frac{1}{\Delta} B$ is an isomorphism from $M(\Delta B^{-1})$ to M(I) and from $M(\Delta I)$ to M(B). It follows that $\frac{1}{\Delta} B$ takes congruence classes in G into congruence classes in G' and that this mapping from G to G' is an isomorphism.

We now take the isomorphism one step further. Smith [25] has shown that by performing certain elementary row and column operations on matrix B one obtains a diagonal matrix $\bar{B}$,

$$\bar{B} = \begin{bmatrix} \epsilon_1 & & & 0 \\ & \epsilon_2 & & \\ & & \ddots & \\ 0 & & & \epsilon_m \end{bmatrix}$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$ are positive integers, $\epsilon_1 \cdot \epsilon_2 \cdot \cdots \epsilon_m = \Delta$, and $\epsilon_i | \epsilon_{i+1}$ for $i = 1,2, \ldots, m-1$. Matrix $\bar{B}$ may be expressed in terms of matrix B as follows,

$$\bar{B} = R\ B\ C$$

where R and C are integer $m \times m$ matrices with $|\det R| = |\det C| = 1$.

We show that the group $G' = \frac{M(I)}{M(B)}$, is isomorphic to the group $G'' = \frac{M(I)}{M(\bar{B})}$. Since $|\det R| = 1$, R is an isomorphism from M(I) to M(I). Moreover, R is an isomorphism from M(B) to M(RB). But $M(RB) = M(\bar{B})$, which can be shown as follows. Let $y \in M(\bar{B})$. Then $y = RBCx = RB(Cx)$, for some integer m-vector x. Thus $y \in M(RB)$ and $M(\bar{B}) \subseteq M(RB)$. Next let $y \in M(RB)$. Then $y = RBx = RBC(C^{-1}x) = \bar{B}(C^{-1}x)$, for some integer m-vector x. Since $|\det C| = 1$, $C^{-1}x$ is an integer m-vector and therefore $y \in M(\bar{B})$. Thus $M(RB) \subseteq M(\bar{B})$, and $M(RB) = M(\bar{B})$. It follows that R is an isomorphism from M(B) to $M(\bar{\bar{B}})$. Furthermore it follows that R takes congruence classes in G' into congruence classes in G", and this mapping from G' to G" is an isomorphism.

The structure of $G'' = \frac{M(I)}{M(\bar{\bar{B}})}$ is apparent. If we let J_k denote the group of integers $\{0, 1, 2, \ldots, k-1\}$ under addition modulo k, then it is readily seen that G" is isomorphic to a direct product group H, where

$$H = J_{\epsilon_1} \times J_{\epsilon_2} \times \cdots \times J_{\epsilon_m}.$$

The isomorphism from G" to H maps a congruence class in G" into the integer m-tuple obtained by taking any representative member of the class and reducing its i^{th} component modulo ϵ_i, for $i = 1, 2, \ldots, m$. The order of H is $\epsilon_1 \cdot \epsilon_2 \cdot \ldots \cdot \epsilon_m = \Delta$. Although we have expressed H as an m-fold direct product, in many cases several of the ϵ_i's have value 1, in which case the corresponding components of the direct product contribute nothing to the structure of H and can be omitted.

The following diagram summarizes the sequence of isomorphic groups we have been discussing.

$$\underset{G}{\frac{M(\Delta B^{-1})}{M(\Delta I)}} \xrightarrow[\cong]{\frac{1}{\Delta}B} \underset{G'}{\frac{M(I)}{M(B)}} \xrightarrow[\cong]{R} \underset{G''}{\frac{M(I)}{M(\bar{\bar{B}})}} \xrightarrow[\cong]{\text{reduction modulo } (\epsilon_1, \epsilon_2, \ldots, \epsilon_m)} \underset{H}{J_{\epsilon_1} \times J_{\epsilon_2} \times \cdots \times J_{\epsilon_m}}$$

We began with the equation (3-6) over the group G as a means of characterizing the members of M(B). We may now replace that equation with an equation over the group H. Letting ϖ denote the isomorphism from G to H, we replace equation (3-6) with

$$\sum_{i=1}^{m} b_i h_i = 0 \tag{3-7}$$

where

$$h_i = \varpi(\beta_i) = \frac{1}{\Delta} RB(\Delta B^{-1})_i \mod \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_m \end{bmatrix}$$

$$= R_i \mod \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_m \end{bmatrix}$$

(The subscript i on a matrix denotes the i^{th} column of that matrix.) Equation (3-7) is the desired characterization of M(B). An integer m-vector b is in M(B) iff b satisfies equation (3-7).

Now we are ready to characterize T(A), the group of all m-vectors b for which the constraints (3-2) and (3-3), with matrix A fixed, define a totally integral convex polyhedron. We recall from Lemma 3-3 that

$$T(A) = \bigcap_{i=1}^{k} M(B_i)$$

where B_i, for $i = 1, 2, \ldots, k$, are all the bases of $[A \,\vdots\, I]$.

For each basis B_i we have seen that $M(B_i)$ may be characterized as the set of integer solutions to an equation

$$\sum_{j=1}^{m} b_j \, h_{ij} = 0$$

over a finite Abelian group H_i of order $\Delta_i = |\det B_i|$. It follows that T(A) may be characterized as the set of integer solutions to k simultaneous equations,

$$\sum_{j=1}^{m} b_j \, h_{ij} = 0, \quad \text{for} \quad i = 1, 2, \ldots, k$$

over the finite Abelian groups $H_1, H_2, \ldots, H_k$. Equivalently, T(A) may be characterized as the set of integer solutions to a single equation,

$$\sum_{j=1}^{m} b_j \, k_j = 0$$

over a finite Abelian group K, where

$$K = H_1 \times H_2 \times \cdots \times H_k$$

and

$$k_j = (h_{1j}, h_{2j}, \ldots, h_{kj}), \quad \text{for} \quad j = 1, 2, \ldots, m.$$

The order of K is $\Delta_1 \cdot \Delta_2 \cdot \ldots \cdot \Delta_k$.

We summarize these results in the following theorem.

Theorem 3-1

Given a fixed integer $m \times n$ matrix A, the set T(A) of all integer m-vectors b for which the constraints

$$[A \mid I] \begin{bmatrix} x \\ \hline z \end{bmatrix} = b$$

$$x \geq 0, \quad z \geq 0$$

define a totally integral convex polyhedron is equal to the set of integer m-vectors b which satisfy an equation

$$\sum_{j=1}^{m} b_j \, k_j = 0$$

over a finite Abelian group K derived from all bases $B_1, B_2, \ldots, B_k$ of $[A \mid I]$. The order of K is $\Delta_1 \cdot \Delta_2 \cdot \ldots \cdot \Delta_k$, where

$$\Delta_i = |\det B_i|. \qquad \square$$

One conclusion we may draw from Theorem 3-1 is that T(A) never consists of just the all zero m-vector. If we let $o(k_j)$ denote the order of k_j in the group K, and if we let $\emptyset$ be the diagonal $m \times m$ matrix,

$$\emptyset = \begin{bmatrix} o(k_1) & & & 0 \\ & o(k_2) & & \\ & & \ddots & \\ 0 & & & o(k_m) \end{bmatrix}$$

we see that

$$M(\emptyset) \subseteq T(A).$$

Thus $T(A)$ contains a countably infinite number of m-vectors.

In terms of convex polyhedra having the totally integral property, these results are somewhat surprising. They tell us that for any matrix A there exist a countably infinite number of right hand side vectors b which give rise to totally integral polyhedra.

CHAPTER 4

TRANSFORMATION OF (n,k)-POLYHEDRA INTO (k,k)-POLYHEDRA

The general problem of integralizing convex (n,k)-polyhedra may be approached inductively with respect to one of the parameters n or k. Recall that an (n,k)-polyhedron is defined by $Ax \leq b$, where A is an $m \times n$ matrix of rank k, $1 \leq k \leq n$. In this chapter we show that the class of (n,k)-polyhedra for fixed k and arbitrary n, $n \geq k$, have a great deal in common with respect to integralization. In particular we show that an (n,k)-polyhedron P_n, where $n \geq k$, may be transformed into a (k,k)-polyhedron P_k by a linear transformation $T\colon\ R^n \to R^k$ which preserves integralization.

In order to explain what we mean when we say that T preserves integralization, we refer to the diagram below. If we let ' denote

$$\begin{tikzcd} P_n \arrow[r, "T"] \arrow[d, "'"'] & P_k \arrow[d, "'"] \\ P_n' \arrow[r, "T"'] & P_k' \end{tikzcd}$$

the operation of integralization, then this diagram expresses the fact that

$$T(P_n') = T(P_n)'.$$

Thus in order to integralize P_n to obtain P_n', we may instead integralize P_k to obtain P_k' and then use the inverse transformation T^{-1} to obtain P_n'. (A word of explanation concerning T^{-1} is in order. As we shall see, T is a many-to-one transformation. Thus it does not make sense to talk about $T^{-1}(y)$, $y \in R^k$, as being a single real point $x \in R^n$. Rather, by $T^{-1}(y)$ we shall mean

$$T^{-1}(y) = \{x \in R^n \mid T(x) = y\}.$$

With this interpretation of T^{-1} we may say that $T^{-1}(P_k') = P_n'$.)

Thus the purpose of this chapter is to show that the general problem of integralizing (n,k)-polyhedra reduces (somewhat) to the problem of integralizing (k,k)-polyhedra. In the first section of this chapter we show how to obtain a suitable transformation T for a particular (n,k)-polyhedron P_n. In the second section we show that such a transformation preserves integralization. And in the third section we begin our inductive approach to integralization by examining the simple case, k = 1.

4.1 Existence of a Suitable Transformation

Let P_n be an (n,k)-polyhedron defined by $Ax \leq b$, where A is an integer $m \times n$ matrix of rank k, and b is an integer m-vector. Any integer $k \times n$ matrix T of rank k whose rows span the row space of A, when viewed as a linear transformation from R^n to R^k, transforms P_n into a (k,k)-polyhedron P_k defined by $Cy \leq b$, where $y = Tx$ and C is an $m \times k$ matrix (not necessarily integer) of rank k defined by the matrix equation A = CT. However, we require a matrix T having two additional properties:

(i) gcd [T] = 1

(ii) every row of A is an integer combination or rows of T.

In the following section we show that a matrix T having these properties preserves integralization. In this section we are concerned with the existence of such a matrix T.

The conditions we have imposed on T are dependent only on the matrix A and are independent of the vector b. The following lemma assures us that a matrix T exists having the required properties.

Lemma 4-1

If A is an integer $m \times n$ matrix of rank k then there exists an integer $k \times n$ matrix T of rank k such that (i) gcd $[T] = 1$ and (ii) every row of A is an integer combination of rows of T.

Proof

Let U be any $k \times n$ submatrix of A having rank k. According to Jacobson [18], U may be written as

$$U = R\,\bar{U}\,C$$

where R is an integer $k \times k$ matrix with $\det R = \pm 1$, C is an integer $n \times n$ matrix with $\det C = \pm 1$, and

$$\bar{U} = [U' \,\vdots\, 0]$$

where U' is a diagonal $k \times k$ matrix and 0 is the $k \times (n-k)$ zero matrix. Let $\bar{T} = [I \,\vdots\, 0]$, where I is the $k \times k$ identity matrix and let

$$T = R\,\bar{T}\,C. \tag{4-1}$$

It is clear that gcd $[\bar{T}] = 1$. Jacobson [18] shows that gcd $[T] =$ gcd $[\bar{T}] = 1$.

We now show that every row of A is an integer combination of rows of T. Let a be any row of A. Since A has rank k and U is a $k \times n$ submatrix of A having rank k, the rows of U span the row space of A. The row a may be written uniquely as a linear combination of the rows of U,

$$a = cU$$

where c is a rational k-vector (row). Then we may write

$$a = cU = c(R\bar{U}C) = cR(U'\bar{T})C = c(RU'R^{-1})\ (R\bar{T}C)$$
$$= c'T$$

where c' is a rational k vector (row) given by $c' = c(RU'R^{-1})$. We show that c' must be an integer k-vector.

We may write $c' = \lfloor c' \rfloor + \langle c' \rangle$ where $\lfloor c' \rfloor$ denotes the greatest integer function applied componentwise to c' and $\langle c' \rangle$ is the vector of fractional parts of c'. Each component of $\langle c' \rangle$ is a non-negative rational number less than 1. Now, since $a = c'T = \lfloor c' \rfloor\ T + \langle c' \rangle\ T$ and $\lfloor c' \rfloor\ T$ is an integer n-vector, we see that $\langle c' \rangle\ T$ is also an integer n-vector. We shall deduce from this that $\langle c' \rangle = 0$.

We represent $\langle c' \rangle$ uniquely by

$$\langle c' \rangle = \frac{1}{d}\ (e_1, e_2, \ldots, e_k) \tag{4-2}$$

where d is a positive integer, $e_1, e_2, \ldots, e_k$ are non-negative integers, $0 \leq e_i < d$, and $\gcd(e_1, e_2, \ldots, e_k, d) = 1$. Let S be a subset of k column indices of T such that the $k \times k$ submatrix T^S comprised of the columns identified by S has non-zero determinant. Since $\langle c' \rangle T$ is an integer m-vector we see that $\langle c' \rangle T^S$ is an integer k vector, say a'. Solving the system of equations $a' = \langle c' \rangle T^S$ for $\langle c' \rangle$ we find that $\langle c' \rangle$ has the form $\frac{1}{|\det T^S|}(f_1, f_2, \ldots, f_k)$, where $f_1, f_2, \ldots, f_k$ are integers. Comparing this with (4-2) we conclude that $d \mid \det T^S$. Similarly, we may show that d divides all $k \times k$ minors of T. Thus $d \mid \gcd[T] = 1$, and $d = 1$. It follows that $e_1 = e_2 = \ldots = e_k = 0$ and $\langle c' \rangle = 0$. $\square$

4.2 The Transformation Preserves Integralization

The transformation T whose existence is guaranteed by Lemma 4-1 transforms the (n,k)-polyhedron P_n defined by $Ax \leq b$ into the (k,k)-polyhedron P_k defined by $Cy \leq b$, where $y = Tx$ and $A = CT$. Property (ii) of T insures that C is an integer matrix. We shall write $P_k = T(P_n)$ and $P_n = T^{-1}(P_k)$. In general, for a subset $Y \subseteq R^k$, $T^{-1}(Y)$ will have the meaning

$$T^{-1}(Y) = \{x \in R^n \mid Tx \in Y\}.$$

We argue next that T takes faces of P_n into faces of P_k. Let $(P_n)_S$ be an arbitrary face of P_n,

$$(P_n)_S = \{x \in P_n \mid A_S x = b_S\}$$

and let $(P_k)_S$ be the face

$$(P_k)_S = \{y \in P_k \mid C_S\, y = b_S\}.$$

Since $A = CT$ we see that $A_S = C_S T$. It follows that $T((P_n)_S) = (P_k)_S$. It also follows that $T^{-1}((P_k)_S) = (P_n)_S$. The minimal faces of P_n are (n-k)-flats in R^n. Their images under T are minimal faces of P_k, 0-flats in R^k.

Let ' denote the operation of integralization. Suppose we have computed P_k', an integral (k,k)-polyhedron defined by $C'y \leq b'$, where C' is an integer $m' \times k$ matrix of rank k and b' is an integer m'-vector. Then $T^{-1}(P_k')$ is an (n,k)-polyhedron defined by $A'x \leq b'$, where A' is an integer $m' \times n$ matrix given by $A' = C'T$. The following lemma asserts that the transformation T preserves integralization.

Lemma 4-2

$$P_n' = T^{-1}(P_k')$$

Proof

We show that $T^{-1}(P_k')$ is integral and $I(T^{-1}(P_k')) = I(P_n)$. It will follow from Lemma 1-1 that $T^{-1}(P_k') = P_n'$.

We first show that $T^{-1}(P_k')$ is integral. Every minimal face of $T^{-1}(P_k')$ is the image of a minimal face of P_k' under T^{-1} and conversely. Minimal faces of P_k' are vertices. Since P_k' is integral, its vertices are integer points in R^k. Let y^0 be any vertex of P_k'. Then $T^{-1}(y^0)$ is a minimal face of $T^{-1}(P_k')$. $T^{-1}(y^0)$ is the (n-k)-flat defined by,

$$T^{-1}(y^0) = \{x \in R^n \mid Tx = y^0\}.$$

Here we make use of property (i) of T, namely gcd [T] = 1. It follows that

$$\gcd [T] = \gcd [T \mid y^0]$$

and, by Theorem 2-1, $T^{-1}(y^0)$ is an integral flat. It follows that every minimal face of $T^{-1}(P_k')$ is integral, and by Corollary 2-4, $T^{-1}(P_k')$ is integral.

We now show that $I(T^{-1}(P_k')) = I(P_n)$. Since $P_k' \subseteq P_k$ and $T^{-1}(P_k) = P_n$ we have $T^{-1}(P_k') \subseteq P_n$ and $I(T^{-1}(P_k')) \subseteq I(P_n)$. Consider the set difference $I(P_n) - I(T^{-1}(P_k'))$. Assume there is an integer point x in this set difference. Since $x \in I(P_n)$, $Tx = y$ is an integer point in P_k, hence in P_k'. But then $x \in I(T^{-1}(P_k'))$ which contradicts our assumption. We conclude that $I(P_n) - I(T^{-1}(P_k'))$ is empty. Thus $I(T^{-1}(P_k')) = I(P_n)$.

Having shown that $T^{-1}(P_k')$ is integral and that $I(T^{-1}(P_k')) = I(P_n)$, it follows from Lemma 1-1 that $T^{-1}(P_k') = P_n'$. □

Summarizing these results, we have seen that given a system of linear inequalities $Ax \leq b$ defining an (n,k)-polyhedron P_n, there exists a transformation T which transforms P_n into the (k,k)-polyhedron P_k defined by $Cy \leq b$, where $y = Tx$ and $A = CT$. Furthermore, if we integralize P_k to obtain P'_k defined by $C'y \leq b'$, then $T^{-1}(P'_k)$ defined by $A'x \leq b'$, where $A' = C'T$, is equal to P'_n. Thus, without loss of generality, we may restrict ourselves to the problem of integralizing (k,k)-polyhedra.

We examine the case $k=1$ in the following section to illustrate the transformation and integralization of $(n-1)$-polyhedra.

4.3 Transformation and Integralization of (n,1)Polyhedra

Let P_n be an $(n,1)$-polyhedron defined by $Ax \leq b$, where A is an integer $m \times n$ matrix of rank 1, and b is an integer m-vector. Let $T = \frac{1}{\Delta_j} a_j$, where a_j is any non-zero row of A and $\Delta_j = \gcd [a_j]$. Then $\gcd [T] = 1$. Moreover, every row a_i of A, $i = 1, 2, \ldots, m$, is an integer multiple of T, $a_i = c_i T$, where c_i is an integer equal to $\pm \gcd [a_i]$. Thus T satisfies the conditions for an integralization preserving transformation.

$T(P_n)$ is the $(1,1)$-polyhedron P_1 defined by $Cy \leq b$, where $y = Tx$ and C is the integer $m \times 1$ matrix defined by $A = CT$. According to Lemma 4-2, $T^{-1}(P'_1) = P'_n$.

The problem of integralizing P_1 is easily solved. Consider a single half-line τ_i defined by $c_i y \leq b_i$, $c_i \neq 0$. We integralize τ_i as follows. Let $\bar{\tau}_i$ be the half-line defined by

$$y \leq \left\lfloor \frac{b_i}{c_i} \right\rfloor \qquad \text{if } c_i > 0$$

$$-y \leq \left\lfloor \frac{b_i}{-c_i} \right\rfloor \qquad \text{if } c_i < 0.$$

The boundary point of $\bar{\tau}_i$ is an integer so that, by Theorem 2-2, $\bar{\tau}_i$ is integral. Furthermore, $I(\bar{\tau}_i) = I(\tau_i)$. It follows from Lemma 1-1 that $\bar{\tau}_i = \tau_i' = H(I(\tau_i))$.

Among all those inequalities $c_i y \leq b_i$ for which $c_i > 0$, if any, let the j^{th} inequality be one for which the ratio $\frac{b_i}{c_i}$ is smallest. Similarly, among all those inequalities $c_i y \leq b_i$ for which $c_i < 0$, if any, let the k^{th} inequality be one for which the ratio $\frac{b_i}{-c_i}$ is smallest. Let $\bar{P}_1$ be defined by the following system of at least one and at most two inequalities,

$$y \leq \left\lfloor \frac{b_j}{c_j} \right\rfloor$$

$$-y \leq \left\lfloor \frac{b_k}{-c_k} \right\rfloor .$$

We see that $I(\bar{P}_1) = I(P_1)$. If $-\left\lfloor \frac{b_k}{-c_k} \right\rfloor > \left\lfloor \frac{b_j}{c_j} \right\rfloor$ then $I(\bar{P}_1) = I(P_1) = \varphi$, where φ denotes the empty set. Thus $\bar{P}_1 = P_1' = \varphi$. Otherwise, by Corollary 2-4, $\bar{P}_1$ is integral. It follows from Lemma 1-1 that $\bar{P}_1 = P_1'$.

$T^{-1}(P_1')$ is an $(n,1)$-polyhedron defined by the following system of at least one and at most two inequalities,

$$Tx \le \left\lfloor \frac{b_j}{c_j} \right\rfloor$$

$$-Tx \le \left\lfloor \frac{b_k}{-c_k} \right\rfloor .$$

By Lemma 4-2, $T^{-1}(P_1') = P_n'$.

<u>Example 4-1</u> Let P_3 be the $(3,1)$-polyhedron defined by

$$\begin{bmatrix} -8 & 4 & 12 \\ 12 & -6 & -18 \\ -4 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} 13 \\ 7 \\ 9 \end{bmatrix}$$

Let $T = \frac{1}{4}\ [-8 \quad 4 \quad 12] = [-2 \quad 1 \quad 3]$. Then $P_1 = T(P_3)$ is the $(1,1)$-polyhedron defined by,

$$\begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix} y \le \begin{bmatrix} 13 \\ 7 \\ 9 \end{bmatrix} .$$

We integralize P_1 and obtain the (1,1)-polyhedron P_1' defined by,

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} y \le \begin{bmatrix} 3 \\ 1 \end{bmatrix} .$$

Then $T^{-1}(P_1')$ is the (3,1)-polyhedron defined by

$$\begin{bmatrix} -2 & 1 & 3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and $T^{-1}(P_1') = P_3'$. $\square$

$$\frac{1}{d_i} \; a^i \cdot x \leq \left\lfloor \frac{b_i}{d_i} \right\rfloor$$

for $i = 1, 2$. Since $I(\tau_i') = I(\tau_i)$ we have

$$I(P) = I(\tau_1 \cap \tau_2) = I(\tau_1' \cap \tau_2')$$

so that P may be replaced by $\tau_1' \cap \tau_2'$.

In the following we assume that this first integralization step has been performed on P. That is, we assume that P is defined by $Ax \leq b$ where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

and

$$\gcd(a_{11}, a_{12}) = 1$$

$$\gcd(a_{21}, a_{22}) = 1.$$

Let $\Delta = \det A$. We assume that the order of the two inequalities $Ax \leq b$ is such that $\Delta > 0$.

Our goal is to generate the system of inequalities $A'x \leq b'$ defining P'. We accomplish this by generating all normal vectors and vertices of I(P). In the following we define the notion of a normal vector of I(P) and a vertex of I(P). We then introduce a transformation of co-ordinates which makes the properties of normal vectors more accessible. In particular, it allows us to identity images of normal vectors as being atoms of a certain partially ordered set.

We then develop a two-dimensional generalization of the division theorem for integers and show how it is used to generate the finite set of these atoms. We then describe a second level two-dimensional division process involving these atoms which, together with the inverse transformation, generates all normal vectors and vertices of I(P). These lead directly to the system $A'x \leq b'$ defining P'.

5.1 Normal Vectors and Vertices

Let $x^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \end{pmatrix}$ be an integer point in I(P). We shall say that x^0 is a boundary point of I(P) iff there exists a non-zero integer vector $\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$ such that

$$\sigma \cdot (x - x^0) \leq 0 \quad \text{for all } x \in I(P)$$

where . denotes vector inner product. We shall say that σ is a support vector of I(P) at x^0. The inequality $\sigma \cdot (x-x^0) \leq 0$ defines a supporting half-plane for I(P).

We shall say that a boundary point of I(P) is a vertex of I(P) iff there exist two linearly independent support vectors of I(P) at x^0.

It is a straightforward matter to compute some of the boundary points of $I(P)$. To do so we make use of some basic results in number theory.

Let π be an integral line defined by $a \cdot x = b$, where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is an integer vector, b is an integer, and $\gcd(a_1, a_2) = 1$. An integer solution x^0 to $a \cdot x = b$ may be determined as follows. The Euclidean algorithm is used to compute a (non-unique) integer vector $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ such that $a \cdot r = 1$. Then $x^0 = br$ is an integer solution to $a \cdot x = b$, and $a \cdot x = b$ may be written as $a \cdot (x - x^0) = 0$.

The following lemma is stated without proof. (For a proof see, for example, Niven and Zuckerman [21].) For convenience in stating it, we let

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

<u>Lemma 5-1</u>

If π is a line defined by $a \cdot (x - x^0) = 0$ where $\gcd(a_1, a_2) = 1$, then $I(\pi)$ is the set of integer points expressible as $x^0 + kTa$, where k is an integer. $\square$

Consider the integral boundary lines π_1 and π_2 of the integral half-planes τ_1 and τ_2 defining P. π_1 and π_2 are defined by $a^1 \cdot x = b_1$

and $a^2 \cdot x = b_2$ respectively, where $a^i = \begin{pmatrix} a_{i1} \\ a_{i2} \end{pmatrix}$ and gcd $(a_{i1}, a_{i2}) = 1$, for $i = 1,2$. The Euclidean algorithm gives two integer points x^1 and x^2 in π_1 and π_2 respectively. The defining equation for π_i may then be written as $a^i \cdot (x - x^i) = 0$, for $i = 1,2$. Lemma 5-1 tells us that $I(\pi_i)$ is the set of integer points expressible as $x^i + k_i Ta^i$, where k_i is an integer.

Certain integer points in $I(\pi_1)$ satisfy $a^2 \cdot x \le b_2$ and thus are integer points in $I(P)$. It is easy to show that $I(\pi_1) \cap I(P)$ is the set of integer points expressible as $x^1 - k_1 Ta^1$, where k_1 is an integer and

$$k_1 \le \left\lfloor \frac{b_2 - a^2 \cdot x^1}{\Delta} \right\rfloor \quad .$$

Similarly, $I(\pi_2) \cap I(P)$ is the set of integer points expressible as $x^2 + k_2 Ta^2$, where k_2 is an integer and

$$k_2 \le \left\lfloor \frac{b_1 - a^1 \cdot x^2}{\Delta} \right\rfloor \quad .$$

Both $I(\pi_1) \cap I(P)$ and $I(\pi_2) \cap I(P)$ are sets of boundary points of $I(P)$.

If x^0 is a boundary point of $I(P)$ then we shall say that a non-zero integer vector σ is a <u>normal vector of I(P)</u> at x^0 iff:

$$\left.\begin{array}{ll} \text{(i)} & \sigma\cdot(x-x^0) \leq 0 \qquad \text{for all } x \in I(P) \\ \text{(ii)} & x^0 \pm T\sigma \in I(P) \\ \text{(iii)} & \gcd(\sigma_1, \sigma_2) = 1. \end{array}\right\} \qquad (5\text{-}1)$$

The inequality $\sigma\cdot(x-x^0) \leq 0$ defines a half-plane τ which is a supporting half-plane for I(P). The boundary line π defined by $\sigma.(x-x^0) = 0$ is a supporting line for I(P). π contains at least two integer points, x^0 and either $x^0 + T\sigma$ or $x^0 - T\sigma$, in I(P). The condition gcd $(\sigma_1, \sigma_2) = 1$ is included to provide a canonical form for σ. If this condition were not imposed, then every positive integer scalar multiple $k\sigma$ of a normal vector σ would also be a normal vector, and as we shall see, this would be inconvenient for our purposes.

Two normal vectors of I(P) are evident. For every boundary point x^1 in $I(\pi_1) \cap I(P)$, a^1 is a normal vector of I(P) at x^1. Similarly, for every boundary point x^2 in $I(\pi_2) \cap I(P)$, a^2 is a normal vector of I(P) at x^2.

Since normal vectors are support vectors, we see that a boundary point x^0 of I(P) is a vertex of I(P) if there exist two linearly independent normal vectors of I(P) at x^0.

In the remainder of this chapter we develop a method for generating the system of inequalities defining H(I(P)). The method involves generating all normal vectors and vertices of I(P). Clearly, if x^0 is a vertex of I(P) and σ is a normal vector of I(P) at x^0, then the inequality $\sigma\cdot(x-x^0) \leq 0$ defines a half-plane whose boundary line contains an edge of H(I(P)). It follows that the system of all inequalities of the form $\sigma\cdot(x-x^0) \leq 0$, where x^0 is a vertex of I(P) and σ is a normal vector of I(P) at x^0, defines H(I(P)).

We begin by introducing a transformation of co-ordinates which makes the properties of normal vectors more accessible.

5.2 A Hull-Formation Preserving Transformation

Let us regard the matrix A of the system $Ax \leq b$ defining P as a linear transformation A: $R^2 \to R^2$. For all $x \in R^2$, A: $x \to y$ where $y = Ax$. Since $\Delta \neq 0$, A is a non-singular linear transformation. Thus the inverse transformation A^{-1} exists. A^{-1} is given by

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} .$$

A^{-1} is not an integer matrix in general.

We shall denote the image $A(X)$ of a subset $X \subseteq R^2$ under A by X^* so that

$$X^* = \{y \in R^2 \mid y = Ax,\ x \in X\}.$$

Since

$$P = \{x \in R^2 \mid Ax \leq b\}$$

it follows that

$$P^* = \{y \in R^2 \mid y \leq b\}.$$

Letting $M(A)$ denote the set of all integer combinations of columns of A we see that

$$I(P)^* = \{y \in M(A) \mid y \leq b\}.$$

We claim that the transformation A preserves the operation of forming the convex hull of $I(P)$. This is illustrated in the following diagram.

$$\begin{tikzcd}
I(P) \arrow[r, "A"] \arrow[d] & I(P)^* \arrow[d] \\
H(I(P)) \arrow[r, "A"] & H(I(P)^*)
\end{tikzcd}$$

This diagram expresses the fact that

$$H(I(P))^* = H(I(P)^*)$$

which we prove in the following lemma.

Lemma 5-2

$$H(I(P))^* = H(I(P)^*)$$

Proof

We first show that $H(I(P))^* \subseteq H(I(P)^*)$. Let $y \in H(I(P))^*$. Then

$$y = A(\lambda_0 x^0 + \lambda_1 x^1 + \lambda_2 x^2)$$

where $\lambda_0, \lambda_1, \lambda_2 \geq 0$, $\lambda_0 + \lambda_1 + \lambda_2 = 1$, and $x^0, x^1, x^2 \in I(P)$. We then have

$$y = \lambda_0 Ax^0 + \lambda_1 Ax^1 + \lambda_2 Ax^2$$

$$= \lambda_0 y^0 + \lambda_1 y^1 + \lambda_2 y^2$$

where $y^0, y^1, y^2 \in I(P)^*$. Thus $y \in H(I(P)^*)$.

We next show that $H(I(P)^*) \subseteq H(I(P))^*$. Let $y \in H(I(P)^*)$. Then

$$y = \lambda_0 y^0 + \lambda_1 y^1 + \lambda_2 y^2$$

where $\lambda_0, \lambda_1, \lambda_2 \geq 0$, $\lambda_0 + \lambda_1 + \lambda_2 = 1$, and $y^0, y^1, y^2 \in I(P)^*$.

We then have

$$y = \lambda_0 Ax^0 + \lambda_1 Ax^1 + \lambda_2 Ax^2$$

$$= A(\lambda_0 x^0 + \lambda_1 x^1 + \lambda_2 x^2)$$

where x^0, x^1, $x^2 \in I(P)$. Thus $y \in H(I(P))^*$. □

We should point out that unless $\Delta = \pm 1$, A does not preserve the operation of integralization. That is, $(P')^* \neq (P^*)'$, where $'$ denotes the operation of integralization. In order to see this, consider P^*. P^* is a (2,2)-corner polyhedron defined by $y \leq b$. Since the vertex $y = b$ of P^* is an integer point, P^* is integral. That is, $P^* = (P^*)'$. However, if $\Delta \neq \pm 1$, then it is not true in general that the vertex $x = A^{-1}b$ of P is an integer point, in which case P is not integral. That is $P' \neq P$. It follows that $(P')^* \neq P^* = (P^*)'$.

<u>Example 5-1</u> Let P be the (2,2)-corner polyhedron defined by

$$\begin{bmatrix} 4 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 23 \\ -5 \end{bmatrix}.$$

Fig. 5-1(a) represents portions of P, I(P) and H(I(P)). Fig. 5-1(b) represents portions of P^*, $I(P)^*$ and $H(I(P)^*)$. □

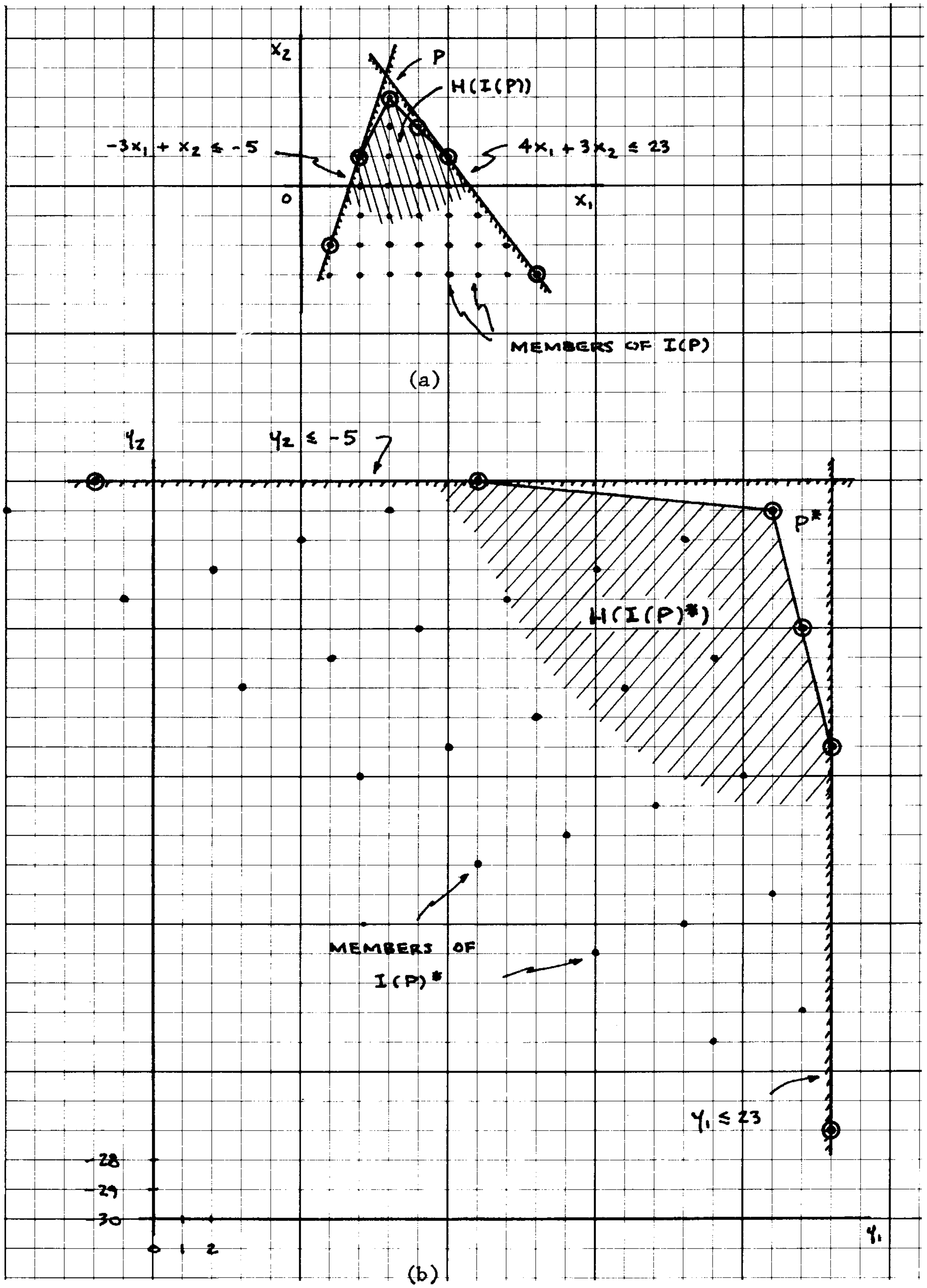

x_2
P
H(I(P))
$-3x_1 + x_2 \leq -5$
$4x_1 + 3x_2 \leq 23$
0
x_1
MEMBERS OF I(P)
(a)
y_2
$y_2 \leq -5$
P*
H(I(P)*)
MEMBERS OF
I(P)*
$y_1 \leq 23$
-28
-29
-30
0
1
2
y_1
(b)

Fig. 5-1

It can be shown that the linear transformation A preserves the face structure of H(I(P)). The image of a face of H(I(P)) is a face of $H(I(P)^*)$ and conversely.

Let x^0 be a boundary point of I(P) and let σ be a normal vector of I(P) at x^0. Then $\sigma\cdot(x-x^0) \leq 0$ for all $x \in I(P)$. In terms of $I(P)^*$ we have

$$\sigma\cdot A^{-1}(y-y^0) \leq 0 \tag{5-2}$$

for all $y \in I(P)^*$, where $y^0 = Ax^0$ is a boundary point of $I(P)^*$. Multiplying (5-2) by $\Delta > 0$ to clear fractions we obtain

$$\sigma\cdot(\Delta A^{-1})(y-y^0) \leq 0$$

or in terms of matrix multiplication only,

$$\sigma^T(\Delta A^{-1})(y-y^0) \leq 0.$$

Letting

$$\gamma^T = \sigma^T(\Delta A^{-1}) \tag{5-3}$$

we have

$$\gamma\cdot(y-y^0) \leq 0$$

for all $y \in I(P)^*$. We shall refer to γ as a <u>normal vector of $I(P)^*$</u> at y^0.

Since normal vectors of $I(P)^*$ play a key role in our development we shall examine their properties in some detail. First of all, we see that a normal vector γ of $I(P)^*$ is related to the corresponding normal vector σ of $I(P)$ by equation (5-3), or equivalently, taking the transpose of each side, by

$$\gamma = (\Delta A^{-1})^T \sigma .$$

Letting $\bar{A} = (\Delta A^{-1})^T$ we have

$$\gamma = \bar{A}\sigma. \tag{5-4}$$

$\bar{A}$ is the cofactor matrix of A and is given by

$$\bar{A} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix} .$$

Letting $M(\bar{A})$ denote the group of all integer combinations of columns of $\bar{A}$ under vector addition, we see from equation (5-4) that $\gamma \in M(\bar{A})$.

The defining conditions (i), (ii), and (iii) in (5-1) for a normal vector σ of $I(P)$ at x^0 may be recast as defining conditions for a normal vector γ of $I(P)^*$ at y^0 as follows. If y^0 is a boundary point of $I(P)^*$ then a non-zero vector $\gamma \in M(\bar{A})$ is a normal vector of $I(P)^*$ at y^0 iff:

$$\left.\begin{array}{ll} \text{(i)} & \gamma\cdot(y-y^0) \leq 0 \quad \text{for all } y \in I(P)^* \\ \text{(ii)} & y^0 \pm T\gamma \subset I(P)^* \\ \text{(iii)} & \lambda\, \gamma \notin M(\bar{A}) \quad \text{for all real } \lambda,\ 0 < \lambda < 1. \end{array}\right\} \tag{5-5}$$

We have already seen that condition (5-5)(i) follows from (5-1)(i). Condition (5-1)(ii) tells us that

$$y^0 \pm AT\sigma \in I(P)^*. \qquad (5\text{-}6)$$

Now $AT = T\bar{A}$, as is easy to verify. Thus $AT\sigma = T\bar{A}\sigma = T\gamma$ and (5-6) becomes condition (5-5)(ii). Finally (5-1)(iii) implies that $\lambda\sigma$ is not an integer vector for all real λ, $0 < \lambda < 1$. Condition (5-5)(iii) follows. Thus, if σ is an integer vector which satisfies (5-1), then $\gamma = \bar{A}\sigma$ is an integer vector which satisfies (5-5). The converse argument can also be made.

<u>Example 5-2</u> Let P be the (2,2)-convex polyhedron defined in Example 5-1. Vertices and associated normal vectors of I(P) and $I(P)^*$ are tabulated below.

I(P)	vertices	$\binom{2}{1}$	$\binom{3}{3}$	$\binom{5}{1}$
	normal vectors	$\binom{-3}{1}, \binom{-2}{1}$	$\binom{-2}{1}, \binom{1}{1}$	$\binom{1}{1}, \binom{4}{3}$
$I(P)^*$	vertices	$\binom{11}{-5}$	$\binom{21}{-6}$	$\binom{23}{-14}$
	normal vectors	$\binom{0}{13}, \binom{1}{10}$	$\binom{1}{10}, \binom{4}{1}$	$\binom{4}{1}, \binom{13}{0}$

Notice that tabulated normal vectors of $I(P)^*$ are images of the corresponding tabulated normal vectors of $I(P)$ under $\bar{A} = \begin{bmatrix} 1 & 3 \\ -3 & 4 \end{bmatrix}$. $\square$

5.3 Normal Vectors as Atoms

In this section we show that normal vectors of $I(P)^*$ are necessarily atoms of a certain partially ordered subset of $M(\bar{A})$. We begin with the following lemma.

Lemma 5-3

If γ is a normal vector of $I(P)^*$ at y^0 then $\gamma \geq 0$.

Proof

According to (5-5)(i),

$$\gamma \cdot (y - y^0) \leq 0 \qquad \text{for all } y \in I(P)^*. \tag{5-7}$$

Recall that $I(P)^*$ is defined by

$$I(P)^* = \{y \in M(A) \mid y \leq b\}.$$

Since $y^0 \in I(P)^*$, $y \in M(A)$ and $y \leq b$. Consider $y^1 = y^0 - \binom{\Delta}{0}$. Since $A^{-1}\binom{\Delta}{0}$ is an integer vector, $\binom{\Delta}{0} \in M(A)$. Thus $y^1 \in M(A)$.

Also, since $y^0 \le b$ and $\Delta > 0$, $y^1 \le b$. Therefore $y^1 \in I(P)^*$. Setting $y = y^1$ in (5-7) we obtain $-\gamma_1 \Delta \le 0$ or

$$\gamma_1 \ge 0.$$

A similar argument using $y^1 = y^0 - \binom{0}{\Delta}$ reveals that

$$\gamma_2 \ge 0.$$

□

A normal vector γ is, by definition, a member of $M(\bar{A})$. If we let $M(\bar{A})^+$ be defined by

$$M(\bar{A})^+ = \{z \in M(\bar{A}) \mid z \ge 0\}$$

then Lemma 5-3 tells us that γ is a member of $M(\bar{A})^+$. As we see next, a much stronger statement about γ can be made.

Consider the partially ordered set $(M(\bar{A})^+, \le)$. The ordering relation $\le$ is, as usual, taken componentwise. The element $0 \in M(\bar{A})^+$ is the least element or universal lower bound.

An <u>atom</u> of $(M(\bar{A})^+, \le)$ is an element $a \in M(\bar{A})$ such that $a \ne 0$ and

$$z \le a \Rightarrow z = 0 \text{ or } z = a$$

for all $z \in M(\bar{A})^+$.

<u>Theorem 5-1</u>

If γ is a normal vector of $I(P)^*$ at y^0 then γ is an atom of $(M(\bar{A})^+, \le)$.

Proof

Let γ be a normal vector of $I(P)^*$ at y^0. By definition, $\gamma \in M(\bar{A})$ and $\gamma \neq 0$. By Lemma 5-3 $\gamma \in M(\bar{A})^+$.

Assume there is a $z \in M(\bar{A})^+$, distinct from 0 and γ such that $z \leq \gamma$. According to (5-5)(ii), $y^0 \pm T\gamma \in I(P)^*$. We show that $y^0 \pm Tz$ and $y^0 \pm T(\gamma - z)$ are also members of $I(P)^*$ and use (5-5)(i) to conclude that $\gamma.(\pm Tz) = 0$. It will follow that $z = \lambda\gamma$, for some real λ, $0 < \lambda < 1$, which contradicts (5-5)(iii).

We first show that $y^0 \pm Tz \in I(P)^*$. Since $z \in M(\bar{A})$ and $T\bar{A} = AT$, it follows that $Tz \in M(A)$. Since $y^0 \in M(A)$, $y^0 \pm Tz \in M(A)$. Now since $y^0 \pm T\gamma \in I(P)^*$ we have $y^0 \pm T\gamma \leq b$. Then since $0 \leq z \leq \gamma$ we have $y^0 \pm Tz \leq b$. Thus $y^0 \pm Tz \in I(P)^*$.

We next show that $y^0 \pm T(\gamma - z) \in I(P)^*$. Since $(\gamma - z) \in M(\bar{A})$ and $T\bar{A} = AT$, it follows that $T(\gamma - z) \in M(A)$. Since $y^0 \in M(A)$ we have $y^0 \pm T(\gamma - z) \in M(A)$. Again $y^0 \pm T\gamma \leq b$. Since $0 \leq (\gamma - z) \leq \gamma$ we have $y^0 \pm T(\gamma - z) \leq b$. Thus $y^0 \pm T(\gamma - z) \in I(P)^*$.

Now we argue that $\gamma \cdot (\pm Tz) = 0$. Setting $y = y^0 \pm Tz$ in (5-5)(i) we obtain $\gamma \cdot (\pm Tz) \leq 0$. Setting $y = y^0 \pm T(\gamma - z)$ in (5-5)(i) we obtain $\gamma \cdot (\pm T(\gamma - z)) \leq 0$. Since $\gamma \cdot (\pm T\gamma) = 0$, this gives $-\gamma \cdot (\pm Tz) \leq 0$. Thus $\gamma \cdot (\pm Tz) = 0$.

Finally $\gamma \cdot (\pm Tz) = 0$ implies that $z = \lambda\gamma$ for some real number λ. Furthermore $0 \leq z \leq \gamma$, $z \neq 0$ and $z \neq \gamma$, imply that $0 < \lambda < 1$. But $z = \lambda\gamma$, $0 < \lambda < 1$, contradicts (5-5)(iii). $\square$

Theorem 5-1 tells us that normal vectors of $I(P)^*$ are necessarily atoms of $(M(\bar{A})^+, \leq)$. As we shall see, there are at most $\Delta + 1$ such atoms. In the remaining sections we show how to generate these atoms and then show how they are used to compute $H(I(P)^*)$, and thus $H(I(P))$.

5.4 A Two-Dimensional Division Theorem

Having shown that normal vectors of $I(P)^*$ are atoms of $(M(A)^+, \leq)$, our next objective is to generate these atoms. To this end we present the following theorem, which may be viewed as a generalization of the division theorem for integers. It applies to integer 2-vectors. The notation $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \begin{matrix} > \\ < \end{matrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ means $a_1 > b_1$ and $a_2 < b_2$. All quantities are integers.

Theorem 5-2

Given $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ such that

$$0 \leq \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \begin{matrix} > \\ < \end{matrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} > 0$$

there exist unique q and $\begin{pmatrix} u_3 \\ v_3 \end{pmatrix}$ such that

$$\binom{u_1}{v_1} = q \binom{u_2}{v_2} - \binom{u_3}{v_3}$$

and

$$0 < \binom{u_2}{v_2} \begin{matrix} > \\ < \end{matrix} \binom{u_3}{v_3} \geq 0$$

Proof

We first show that q and $\binom{u_3}{v_3}$ exist.

Let q and u_3 be determined by the division theorem applied to u_1 and u_2,

$$u_1 = q\, u_2 - u_3, \qquad u_2 > u_3 \geq 0.$$

Since $u_1 > u_2 > 0$, it follows that $q > 1$.
Then v_3 is given by

$$v_3 = q\, v_2 - v_1$$

and since $q > 1$ and $0 \leq v_1 < v_2$, it follows that $0 < v_2 < v_3$.

We next show that q and $\binom{u_3}{v_3}$ are unique.

Assume

$$\binom{u_1}{v_1} = q \binom{u_2}{v_2} - \binom{u_3}{v_3} \tag{5-8}$$

and

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = q' \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} - \begin{pmatrix} u_3' \\ v_3' \end{pmatrix} \tag{5-9}$$

where

$$0 < \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \begin{matrix} > \\ < \end{matrix} \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} \geq 0 \quad \text{and} \quad 0 < \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \begin{matrix} > \\ < \end{matrix} \begin{pmatrix} u_3' \\ v_3' \end{pmatrix} \geq 0 \, . \tag{5-10}$$

Subtracting (5-9) from (5-8) gives

$$(q - q') \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} - \begin{pmatrix} u_3' \\ v_3' \end{pmatrix}. \tag{5-11}$$

Now from (5-10),

$$u_2 > u_3 \geq 0 \tag{5-12}$$

and

$$0 \geq -u_3' > -u_2 \, . \tag{5-13}$$

Adding (5-12) and (5-13) gives

$$u_2 > u_3 - u_3' > -u_2$$

which, together with the following equation from (5-11) ,

$$(q - q') \, u_2 = u_3 - u_3'$$

implies

$$q = q'.$$

This, together with (5-11) implies

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_3' \\ v_3' \end{pmatrix} .$$

□

Given two ordered pairs of integers $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$ and $\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ such that $0 \le \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \begin{matrix} > \\ < \end{matrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \begin{matrix} > 0 \\ \end{matrix}$ successive application of Theorem 5-2 yields a sequence of ordered pairs

$$0 \le \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \begin{matrix} > \\ < \end{matrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \begin{matrix} > \\ < \end{matrix} \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} \begin{matrix} > \\ < \end{matrix} \cdots \begin{matrix} > \\ < \end{matrix} \begin{pmatrix} u_{k-2} \\ v_{k-2} \end{pmatrix} \begin{matrix} > \\ < \end{matrix} \begin{pmatrix} u_{k-1} \\ v_{k-1} \end{pmatrix} \begin{matrix} > \\ < \end{matrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix} \begin{matrix} = 0 \\ \end{matrix} \tag{5-14}$$

The sequence (5-14) terminates when $u_k = 0$, which occurs for some finite k since $u_1, u_2, \ldots, u_k$ is a monotonically decreasing sequence and u_1 is a positive integer.

There is some symmetry in the statement of Theorem 5-2 which allows us to apply the theorem in the reverse direction. If we display the ordering relations among the components of $\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$ and $\begin{pmatrix} u_3 \\ v_3 \end{pmatrix}$ as follows,

$$0 \le \begin{pmatrix} v_3 \\ u_3 \end{pmatrix} \begin{matrix} > \\ < \end{matrix} \begin{pmatrix} v_2 \\ u_2 \end{pmatrix} \begin{matrix} > 0 \\ \end{matrix}$$

we find that we have two ordered pairs to which Theorem 5-2 may be applied producing unique q and $\begin{pmatrix} v_1 \\ u_1 \end{pmatrix}$ such that

$$\begin{pmatrix} v_3 \\ u_3 \end{pmatrix} = q \begin{pmatrix} v_2 \\ u_2 \end{pmatrix} - \begin{pmatrix} v_1 \\ u_1 \end{pmatrix}$$

and

$$0 < \begin{pmatrix} v_2 \\ u_2 \end{pmatrix} \genfrac{}{}{0pt}{}{>}{<} \begin{pmatrix} v_1 \\ u_1 \end{pmatrix} \geq 0$$

<u>Example 5-3</u> Let $\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 16 \\ 3 \end{pmatrix}$, $\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$. Successive application of Theorem 5-2 gives the sequence

$$\begin{pmatrix} 16 \\ 3 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} \begin{pmatrix} 5 \\ 21 \end{pmatrix} \begin{pmatrix} 3 \\ 34 \end{pmatrix} \begin{pmatrix} 1 \\ 47 \end{pmatrix} \begin{pmatrix} 0 \\ 107 \end{pmatrix}.$$

Applying Theorem 5-2 in the reverse direction, we may extend this sequence to the following one:

$$\begin{pmatrix} 107 \\ 0 \end{pmatrix} \begin{pmatrix} 41 \\ 1 \end{pmatrix} \begin{pmatrix} 16 \\ 3 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} \begin{pmatrix} 5 \\ 21 \end{pmatrix} \begin{pmatrix} 3 \\ 34 \end{pmatrix} \begin{pmatrix} 1 \\ 47 \end{pmatrix} \begin{pmatrix} 0 \\ 107 \end{pmatrix} .$$ □

The sequence of first components in (5-14)

$$u_1 > u_2 > \cdots > u_k = 0$$

is one which would be produced by the Euclidean algorithm applied to u_1 and u_2. The set of common divisors of u_1 and u_2 is equal to the set of common divisors of u_i and u_{i+1}, for $i = 1, 2, \ldots, k-1$. In particular, if u_1 and u_2 are relatively prime, then so are u_i and u_{i+1}, for $i = 1, 2, \ldots, k-1$.

The sequence of second components in (5-14)

$$v_k > v_{k-1} > \cdots > v_1 \geq 0$$

is one which would be produced by the Euclidean algorithm applied to v_k and v_{k-1}. Similar remarks about the set of common divisors of v_i and v_{i+1} for $i = 1, 2, \ldots, k-1$ apply here.

Any three consecutive ordered pairs in the sequence (5-14) are related by an equation,

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} = q_i \begin{pmatrix} u_{i+1} \\ v_{i+1} \end{pmatrix} - \begin{pmatrix} u_{i+2} \\ v_{i+2} \end{pmatrix}$$

for $i = 1, 2, \ldots, k-2$. These equations may be used to express any ordered pair in the sequence (5-14) as an integer combination of any two consecutive ordered pairs in the sequence.

<u>Example 5-4</u> In the previous example, $\begin{pmatrix} 16 \\ 3 \end{pmatrix}$ may be expressed as an integer combination of $\begin{pmatrix} 3 \\ 34 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 47 \end{pmatrix}$ as follows,

$$\begin{pmatrix} 16 \\ 3 \end{pmatrix} = 7 \begin{pmatrix} 3 \\ 34 \end{pmatrix} - 5 \begin{pmatrix} 1 \\ 47 \end{pmatrix}.$$

Similarly, $\begin{pmatrix} 3 \\ 34 \end{pmatrix}$ may be expressed as an integer combination of $\begin{pmatrix} 41 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 16 \\ 3 \end{pmatrix}$ as follows,

$$\begin{pmatrix} 3 \\ 34 \end{pmatrix} = -5 \begin{pmatrix} 41 \\ 1 \end{pmatrix} + 13 \begin{pmatrix} 16 \\ 3 \end{pmatrix}.$$

□

5.5 Generation of Atoms

In this section we show how to generate the set of atoms of $(M(\bar{A})^+, \leq)$. Before doing so, we establish some preliminary results concerning $M(\bar{A})$.

Recall that $\bar{A}$ is given by

$$\bar{A} = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

where $\gcd(a_{11}, a_{12}) = 1$ and $\gcd(a_{21}, a_{22}) = 1$. For simplicity in our notation, we let

$$\bar{A} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

where $\alpha_{11} = a_{22}$, $\alpha_{12} = -a_{21}$, $\alpha_{21} = -a_{12}$ and $\alpha_{22} = a_{11}$. Thus $\gcd(\alpha_{11}, \alpha_{12}) = 1$ and $\gcd\ (\alpha_{21}, \alpha_{22}) = 1$.

Let $\alpha^1 = \binom{\alpha_{11}}{\alpha_{12}}$ and $\alpha^2 = \binom{\alpha_{21}}{\alpha_{22}}$. The Euclidean algorithm may be used to compute an $r^1 = \binom{r_{11}}{r_{12}}$ and an $r^2 = \binom{r_{21}}{r_{22}}$ such that $\alpha^1 \cdot r^1 = 1$ and $\alpha^2 \cdot r^2 = 1$. They are not unique. Let g_1 be an integer such that

$$g_1 \equiv \alpha^1 \cdot r^2 \mod \Delta, \quad 0 \le g_1 < \Delta \tag{5-15}$$

and let g_2 be an integer such that

$$g_2 \equiv \alpha^2 \cdot r^1 \mod \Delta, \quad 0 \le g_2 < \Delta \tag{5-16}$$

where $\Delta = \det \bar{A} = \det A > 0$.

Integers g_1 and g_2 are uniquely defined even though vectors r^1 and r^2 are not. To see that g_1 is uniquely defined, let r^2 and $(r^2)'$ be any two vector solutions x to the equation $\alpha^2 \cdot x = 1$. We may write $\alpha^2 \cdot x = 1$ as $\alpha^2 \cdot (x - r^2) = 0$. From Lemma 5-1, $(r^2)' = r^2 + kT\alpha^2$ for some integer k. It follows that $\alpha^1 \cdot (r^2)' \equiv \alpha^1 \cdot r^2 \mod \Delta$. Thus g_1 is uniquely defined by (5-15). A similar argument reveals that g_2 is uniquely defined by (5-16).

<u>Lemma 5-4</u>

$$g_1 \, g_2 \equiv 1 \mod \Delta$$

<u>Proof</u>

$$\begin{aligned} g_1 \, g_2 &\equiv (\alpha^1 \cdot r^2)\,(\alpha^2 \cdot r^1) \mod \Delta \\ &\equiv \left[(\alpha^1 \cdot r^1)\,(\alpha^2 \cdot r^2) - (\alpha^1 \cdot T\alpha^2)\,(r^1 \cdot Tr^2)\right] \mod \Delta \\ &\equiv \left[\,1 - \Delta\,(r^1 \cdot Tr^2)\right] \mod \Delta \\ &\equiv 1 \mod \Delta \end{aligned}$$

□

Now, let $(r^1)'$ and $(r^2)'$ be any two vector solutions to the equations $\alpha^1 \cdot x = 1$ and $\alpha^2 \cdot x = 1$ respectively. Let

$$r^1 = (r^1)' + k_1 T\alpha^1$$

where

$$k_1 = \frac{\alpha^2 \cdot (r^1)' - g_2}{\Delta}$$

and let

$$r^2 = (r^2)' + k_2 T\alpha^2$$

where

$$k_2 = \frac{g_1 - \alpha^1 \cdot (r^2)'}{\Delta} .$$

Then $\alpha^1 \cdot r^1 = 1$, $\alpha^2 \cdot r^2 = 1$, $\alpha^1 \cdot r^2 = g_1$ and $\alpha^2 \cdot r^1 = g_2$. We define matrices C_1 and C_2 as follows:

$$C_1 = \begin{bmatrix} \alpha_{22} & r_{21} \\ -\alpha_{21} & r_{22} \end{bmatrix} \qquad C_2 = \begin{bmatrix} r_{11} & -\alpha_{12} \\ r_{12} & \alpha_{11} \end{bmatrix} .$$

We observe that det $C_1 = 1$ and det $C_2 = 1$. Let $G_1 = \bar{A}C_1$ and $G_2 = \bar{A}C_2$. Then

$$G_1 = \begin{bmatrix} \Delta & g_1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad G_2 = \begin{bmatrix} 1 & 0 \\ g_2 & \Delta \end{bmatrix} . \tag{5-17}$$

Lemma 5-5

$$M(\bar{A}) = M(G_1) = M(G_2)$$

Proof

We prove that $M(\bar{A}) = M(G_1)$. Let $z \in M(\bar{A})$. Then $z = \bar{A}x$ for some integer vector x. It follows that $z = \bar{A}(C_1 C_1^{-1})x = \bar{A}C_1(C_1^{-1}x)$, where C_1^{-1} is an integer matrix because $\det C_1 = 1$, and $C_1^{-1}x$ is an integer vector. Thus $z \in M(\bar{A}C_1) = M(G_1)$ and we have shown that $M(\bar{A}) \subseteq M(G_1)$.

Now let $z \in M(G_1)$. Then $z = G_1x = (\bar{A}C_1)x = \bar{A}(C_1x)$ for some integer vector x. Since C_1x is an integer vector, $z \in M(\bar{A})$. Thus $M(G_1) \subseteq M(\bar{A})$, and we have shown that $M(\bar{A}) = M(G_1)$.

A similar argument gives $M(\bar{A}) = M(G_2)$. $\square$

Lemma 5-6

$$\gcd(\Delta, g_1) = 1 \quad \text{and} \quad \gcd(\Delta, g_2) = 1.$$

Proof

We prove that $\gcd(\Delta, g_1) = 1$. From (5-17) we see that $\begin{pmatrix} 1 \\ g_2 \end{pmatrix} \in M(G_2)$. By Lemma 5-5, $M(G_2) = M(G_1)$, so that $\begin{pmatrix} 1 \\ g_2 \end{pmatrix} \in M(G_1)$. Thus $\begin{pmatrix} 1 \\ g_2 \end{pmatrix} = G_1x$ for some integer vector x. This says that

$$1 = x_1 \Delta + x_2 g_1$$

from which we conclude that $\gcd(\Delta, g_1) \mid 1$. It follows that $\gcd(\Delta, g_1) = 1$.

A similar proof shows that gcd $(\Delta, g_2) = 1$. $\square$

We are now ready to generate the set of atoms of the partially ordered set $(M(A)^+, \leq)$. Suppose we apply our two-dimensional division theorem successively to the columns of G_1. That is, suppose we let

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \Delta \\ 0 \end{pmatrix} \qquad \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ 1 \end{pmatrix}$$

and apply Theorem 5-2 successively to obtain the finite sequence

$$0 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \gtrless \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \gtrless \begin{pmatrix} u_3 \\ v_3 \end{pmatrix} \gtrless \cdots \gtrless \begin{pmatrix} u_{k-2} \\ v_{k-2} \end{pmatrix} \gtrless \begin{pmatrix} u_{k-1} \\ v_{k-1} \end{pmatrix} \gtrless \begin{pmatrix} u_k \\ v_k \end{pmatrix} = 0 \tag{5-18}$$

where $k \geq 2$. We shall show that the set of integer vectors appearing in this sequence is precisely the set of atoms of $(M(\bar{A})^+, \leq)$.

For convenience we let W be the $2 \times k$ matrix,

$$W = \begin{bmatrix} u_1 & u_2 & u_3 & \cdots & u_{k-2} & u_{k-1} & u_k \\ v_1 & v_2 & v_3 & & v_{k-2} & v_{k-1} & v_k \end{bmatrix}$$

whose columns $w^i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$, $i = 1, 2, \ldots, k$, are the integer vectors appearing in the sequence (5-18). Thus

$$W = \left[\, G_1 \;\middle|\; \begin{matrix} u_3 & \cdots & u_{k-2} & u_{k-1} & u_k \\ v_3 & & v_{k-2} & v_{k-1} & v_k \end{matrix} \right].$$

We first argue that every column w^i of W is a member of $M(\bar{A})^+$. As we noted in the previous section, the equations used to derive the columns of W allow us to express every column w^i as an integer combination of the columns of G_1. Thus $w^i \in M(G_1)$. From Lemma 5-5, $w^i \in M(\bar{A})$. Furthermore, since $u_i \geq 0$ and $v_i \geq 0$, $w^i \in M(\bar{A})^+$.

We next show that

$$\begin{bmatrix} u_{k-1} & u_k \\ v_{k-1} & v_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ g_2 & \Delta \end{bmatrix} .$$

We begin by showing that $u_{k-1} = 1$. From Lemma 5-6, gcd $(u_1, u_2) = 1$. Furthermore, as previously noted, gcd (u_i, u_{i+1}) = gcd (u_1, u_2), for $i = 1, 2, \ldots, k-1$. Thus gcd $(u_{k-1}, u_k) = 1$. Since $u_k = 0$ and $u_{k-1} > 0$, it follows that $u_{k-1} = 1$.

We now show that $v_k = \Delta$ by expressing w^k as an integer combination of the columns of G_1. The equations used to derive W are as follows:

$$\begin{aligned} w^1 &= q_1 w^2 - w^3 \\ w^2 &= q_2 w^3 - w^4 \\ &\vdots \\ w^{k-3} &= q_{k-3} w^{k-2} - w^{k-1} \\ w^{k-2} &= q_{k-2} w^{k-1} - w^k . \end{aligned} \tag{5-19}$$

From the last of these equations we have

$$w^k = q_{k-2}\, w^{k-1} - w^{k-2}. \tag{5-20}$$

Since $u_k = 0$ and $u_{k-1} = 1$, we see that $q_{k-2} = u_{k-2}$. We then write (5-20) as

$$w^k = u_{k-2}w^{k-1} - u_{k-1}w^{k-2} . \tag{5-21}$$

Then using (5-21) and equations (5-19) we have

$$\begin{aligned}
w^k &= u_{k-2}(q_{k-3}w^{k-2} - w^{k-3}) - u_{k-1}w^{k-2} \\
&= (q_{k-3}u_{k-2} - u_{k-1})w^{k-2} - u_{k-2}w^{k-3} \\
&= u_{k-3}w^{k-2} - u_{k-2}w^{k-3} \\
&\vdots \\
w^k &= u_i w^{i+1} - u_{i+1}w^i \qquad (5\text{-}22) \\
&\vdots \\
&= u_2w^3 - u_3w^2 \\
&= u_1w^2 - u_2w^1 .
\end{aligned}$$

From the last of these equations we see that

$$\begin{aligned}
v^k &= u_1v_2 - u_2v_1 \\
&= \Delta\cdot 1 - g_1\cdot 0 .
\end{aligned}$$

Thus $v_k = \Delta$.

Finally we argue that $v_{k-1} = g_2$. Since $w^{k-1} \in M(G_1)$ we may write $w^{k-1} = G_1 x$ for some integer vector x. Thus $w^{k-1} = \begin{pmatrix} x_1 \Delta + g_1 x_2 \\ x_2 \end{pmatrix}$, from which we conclude that

$$u_{k-1} \equiv g_1 v_{k-1} \bmod \Delta.$$

Multiplying both sides by g_2 we obtain

$$g_2 u_{k-1} \equiv g_1 g_2 v_{k-1} \bmod \Delta.$$

Using Lemma 5-4 and the fact that $u_{k-1} = 1$, we have

$$v_{k-1} \equiv g_2 \bmod \Delta$$

and this, together with $0 \le v_{k-1} < \Delta$, implies $v_{k-1} = g_2$.

Thus we may write matrix W as

$$W = \left[\; G_1 \;\middle|\; \begin{matrix} u_3 & \cdots & u_{k-2} \\ v_3 & & v_{k-2} \end{matrix} \;\middle|\; G_2 \right].$$

W may also be generated from right to left by starting with the columns of G_2 and successively applying Theorem 5-2 in the reverse direction.

<u>Example 5-5</u> Let

$$\bar{A} = \begin{bmatrix} 103 & -46 \\ -27 & 61 \end{bmatrix}.$$

Then $\Delta = \det \bar{A} = (103)(61) - (-27)(-46) = 5041$. The Euclidean algorithm gives

$$21(103) + 47(-46) = 1$$

$$9(-27) + 4(61) = 1$$

Then

$$\begin{aligned} g_1 &\equiv 9(103) + 4(-46) \bmod 5041 \\ &\equiv 743 \bmod 5041 \end{aligned}$$

and

$$\begin{aligned} g_2 &\equiv 21(-27) + 47(61) \bmod 5041 \\ &\equiv 2300 \bmod 5041. \end{aligned}$$

Thus

$$G_1 = \begin{bmatrix} 5040 & 743 \\ 0 & 1 \end{bmatrix} \qquad G_2 = \begin{bmatrix} 1 & 0 \\ 2300 & 5041 \end{bmatrix}$$

Matrix W is generated using either G_1 or G_2 and Theorem 5-2.

$$W = \left[\begin{array}{cc|ccccccc|cc} 5041 & 743 & 160 & 57 & 11 & 9 & 7 & 5 & 3 & 1 & 0 \\ 0 & 1 & 7 & 34 & 95 & 536 & 977 & 1418 & 1859 & 2300 & 5041 \end{array}\right]$$

□

We require two additional lemmas before we prove that the set of columns of W is the set of atoms of $(M(\bar{A})^+, \leq)$.

Lemma 5-7

Let G_i, $i = 1, 2, \ldots, k-1$, be the 2×2 submatrix of W given by,

$$G_i = \begin{bmatrix} u_i & u_{i+1} \\ v_i & v_{i+1} \end{bmatrix}.$$

Then $M(G_i) = M(\bar{A})$. Furthermore, $\det G_i = \det \bar{A} = \Delta$.

Proof

The equations used to generate the columns of W allow us to express the columns of G_i as integer combinations of the columns of G_1. Thus $M(G_i) \subseteq M(G_1)$. The same equations allow us to express the columns of G_1 as integer combinations of the columns of G_i. Thus $M(G_1) \subseteq M(G_i)$, and we have shown that $M(G_i) = M(G_1)$. By Lemma 5-5, $M(G_i) = M(\bar{A})$.

Furthermore, from equation (5-22) we see that

$$v_k = u_i v_{i+1} - u_{i+1} v_i$$

and since $v_k = \Delta$, we have $\det G_i = \Delta$. $\square$

We define $M^+(W)$ to be the set of all non-negative integer combinations of columns of W,

$$M^+(W) = \{z \in J^2 \mid z = Wx,\ x \in J^k,\ x \geq 0\}.$$

Lemma 5-8

$$M^{+}(W) = M(\bar{A})^{+}$$

Proof

Since each column of W is a member of $M(\bar{A})^{+}$, it follows that every non-negative integer combination of columns of W is a member of $M(\bar{A})^{+}$. Thus $M^{+}(W) \subseteq M(\bar{A})^{+}$.

We now argue that $M(\bar{A})^{+} \subseteq M^{+}(W)$. Let $z \in M(\bar{A})^{+}$. If $z = 0$ then clearly $z \in M^{+}(W)$. Therefore we can assume that $z \neq 0$, that is $z_1 > 0$ or $z_2 > 0$ or both.

Consider the quantities

$$\begin{vmatrix} z_1 & u_i \\ z_2 & v_i \end{vmatrix}$$

for $i = 1, 2, \ldots, k$. For $i = 1$ we have

$$\begin{vmatrix} z_1 & u_1 \\ z_2 & v_1 \end{vmatrix} = \begin{vmatrix} z_1 & \Delta \\ z_2 & 0 \end{vmatrix} = -z_2\,\Delta \leq 0 \tag{5-23}$$

and for $i = k$ we have

$$\begin{vmatrix} z_1 & u_k \\ z_2 & v_k \end{vmatrix} = \begin{vmatrix} z_1 & 0 \\ z_2 & \Delta \end{vmatrix} = z_1\,\Delta \geq 0. \tag{5-24}$$

We claim that

$$\begin{vmatrix} z_1 & u_i \\ z_2 & v_i \end{vmatrix} < \begin{vmatrix} z_1 & u_{i+1} \\ z_2 & v_{i+1} \end{vmatrix} \tag{5-25}$$

for $i = 1, 2, \ldots, k-1$. To prove this we observe that

$$0 \leq v_i < v_{i+1} \quad \text{and} \quad u_i > u_{i+1} \geq 0.$$

(5-25) follows from this and our assumption that $z_1 > 0$ or $z_2 > 0$ or both.

Now, from (5-23), (5-24) and (5-25) we conclude that either:

(i) for some i, $1 \leq i < k$

$$\begin{vmatrix} z_1 & u_i \\ z_2 & v_i \end{vmatrix} < 0 \quad \text{and} \quad \begin{vmatrix} z_1 & u_{i+1} \\ z_2 & v_{i+1} \end{vmatrix} > 0$$

or

(ii) for some i, $1 \leq i \leq k$

$$\begin{vmatrix} z_1 & u_i \\ z_2 & v_i \end{vmatrix} = 0$$

We consider case (i). By Lemma 5-7, $M(G_i) = M(\bar{A})$. Since $z \in M(\bar{A})^+$ it follows that $z \in M(G_i)$ and $z = G_i x$ for some integer vector x. We show that $x_1 > 0$ and $x_2 > 0$. From Lemma 5-7, $\det G_i = \Delta$. Applying Cramer's rule, we solve $z = G_i x$ for x:

$$x_1 = \frac{1}{\Delta} \begin{vmatrix} z_1 & u_{i+1} \\ z_2 & v_{i+1} \end{vmatrix} > 0$$

$$x_2 = \frac{1}{\Delta} \begin{vmatrix} u_i & z_1 \\ v_i & z_2 \end{vmatrix} = -\frac{1}{\Delta} \begin{vmatrix} z_1 & u_i \\ z_2 & v_i \end{vmatrix} > 0$$

Thus z is a non-negative integer combination of columns of G_i.

In case (ii), similar reasoning gives $z = x_1 \binom{u_i}{v_i}$ where x_1 is a positive integer.

Thus we have shown that $z \in M^+(W)$. It follows that $M(\bar{A})^+ \subseteq M^+(W)$. We conclude that $M^+(W) = M(\bar{A})^+$. $\square$

We are now ready to prove the main result of this section.

Theorem 5-3

The set of columns of W is the set of atoms of $(M(\bar{A})^+, \leq)$.

Proof

We first show that every column w^i of W is an atom of $(M(\bar{A})^+, \leq)$. We know that $w^i \in M(A)^+$ and $w^i \neq \mathbf{0}$. Assume that there is a $z \in M(\bar{A})^+$

distinct from 0 and w^i such that $z \leq w^i$. By Lemma 5-8, z may be expressed as

$$z = \sum_{i=1}^{k} c_i w^i$$

where c_i is a non-negative integer, for i = 1, 2, ..., k. If $c_i = 0$, for i = 1, 2, ..., k, then z = 0 contrary to assumption. Thus $c_j > 0$ for some j, $1 \leq j \leq k$, and it follows that $w^j \leq z$. Since $w^j \leq z$ and $z \leq w^i$ it follows that $w^j \leq w^i$, which implies that $w^j = w^i$, because

$$u_1 > u_2 > \cdots > u_k$$

$$v_1 < v_2 < \cdots < v_k.$$

Now we have $w^j \leq z \leq w^i$ and $w^j = w^i$. We conclude that $z = w^i$ which again contradicts our assumption. Therefore no such z exists and w^i is an atom of $(M(\bar{A})^+, \leq)$.

We next show that every atom of $(M(\bar{A})^+, \leq)$ is a column of W. Let a be an atom of $(M(\bar{A})^+, \leq)$. Then $a \neq 0$ and

$$z \leq a \quad \Rightarrow \quad z = 0 \quad \text{or } z = a \tag{5-27}$$

for $z \in M(\bar{A})^+$. By Lemma 5-8, a may be expressed as

$$a = \sum_{i=1}^{k} c_i w^i$$

where c_i is a non-negative integer, for $i = 1, 2, \ldots, k$. Since $a \neq 0$, there is some $c_j > 0$, $1 \leq j \leq k$, in this expression. It follows that $w^j \leq a$. According to (5-27), $w^j = 0$ or $w^j = a$. We know that $w^j \neq 0$. Thus $a = w^j$. $\square$

Matrix W has at most $\Delta + 1$ columns. This follows from the strict ordering,

$$\Delta = u_1 > u_2 > \cdots > u_k = 0$$

$$0 = v_1 < v_2 < \cdots < v_k = \Delta$$

Theorem 5-3 tells us that $(M(\bar{A})^+, \leq)$ has at most $\Delta + 1$ atoms.

5.6 Generation of Normal Vectors and Vertices

We have seen that a normal vector γ of $I(P)^*$ at a vertex y^0 is an atom of the partially ordered set $(M(\bar{A})^+, \leq)$, where $\bar{A}$ is the cofactor matrix of A and $Ax \leq b$ defines P. We have developed a method for generating the matrix W whose columns are the atoms of $(M(\bar{A})^+, \leq)$. In this section we complete our development of an integralization method for (2,2)-corner polyhedra by showing how all normal vectors and vertices of $I(P)^*$, hence of $I(P)$, are generated. Our final product will be a system of inequalities $A'x \le b'$ which defines $P' = H(I(P))$.

We begin by considering the following question. Given a boundary point y^i of $I(P)^*$, how do we tell which columns of W are normal vectors of $I(P)^*$ at y^i?

Let y^i be a boundary point of $I(P)^*$. Since $y^i \in I(P)^*$ we have $y^i \le b$. Let $z^i = b-y^i$. Then $z^i \ge 0$. z^i measures the "slack" of y^i.

Recall that a normal vector γ at y^i satisfies the following conditions:

$$\text{(i)} \quad \gamma\cdot(y-y^i) \le 0 \qquad \text{for all } y \in I(P)^*$$

$$\text{(ii)} \quad y^i \pm T\gamma \in I(P)^*.$$

Using the slack vector z^i, we label two columns of W as shown below:

$$z_2^i$$

$$\Delta = u_1 > \cdots > u_p > \cdots > u_q > \cdots > u_k = 0$$

$$0 = v_1 < \cdots < v_p < \cdots < v_q < \cdots < v_k = \Delta$$

$$z_1^i$$

v_p is the largest element in the lower row of W which is less than or equal to z_1^i. u_q is the largest element in the upper row of W which is less than or equal to z_2^i.

Theorem 5-4

If y^i is a boundary point of $I(P)^*$ then the columns w^p and w^q of W are normal vectors, γ^{i-1} and γ^{i+1} respectively, of $I(P)^*$ at y^0.

Proof

According to Theorems 5-1 and 5-3, normal vectors of $I(P)^*$ at y^i are columns of W.

We first observe that there exists a normal vector γ^{i+1} at y^i such that $y^i - T\gamma^{i+1} \in I(P)^*$. We shall consider all columns w of W such that $y^i - Tw \in I(P)^*$ to be candidates for γ^{i+1} and show that all but w^q fail to satisfy the condition $w\cdot(y-y^i) \leq 0$ for all $y \in I(P)^*$. It will follow by elimination that $\gamma^{i+1} = w^q$.

For all columns w of W, $y^i \pm Tw \in M(A)$. This follows from the following argument. Since $w \in M(\bar{A})^+$, $w = \bar{A}x$ for some integer vector x. Then $Tw = T\bar{A}x = ATx$, so that $Tw \in M(A)$. Since $y^i \in M(A)$, $y^i \pm Tw \in M(A)$. Thus $y^i \pm Tw \in I(P)^*$ iff $y^i \pm Tw \leq b$.

Now, for all columns w^j, where $j < q$, $y^i - Tw^j \not\leq b$ because $y_2^i + u_j > b_2$. For all columns w^j, where $q \leq j \leq k$, $y^i - Tw^j \leq b$ because $y_1^i - v_j \leq b_1$ and $y_2^i + u_j \leq b_2$. Thus $y^i - Tw^j \in I(P)^*$ for $j = q, q+1, \ldots, k$. However, for all column w^j, $j = q+1, q+2, \ldots, k$,

$$w^j \cdot ((y^i - Tw^q) - y^i) = w^j\cdot(-Tw^q) = \begin{vmatrix} u_q & u_j \\ v_q & v_j \end{vmatrix} > 0$$

because $u_q > u_j \geq 0$ and $v_j > v_q \geq 0$. Thus all of these columns fail to satisfy the condition $w^j \cdot (y-y^i) \leq 0$ for all $y \in I(P)^*$. Therefore, by elimination, $\gamma^{i+1} = w^q$.

We next observe that there exists a normal vector γ^{i-1} at y^i such that $y^i + T\gamma^{i-1} \in I(P)^*$. A proof similar to the one above establishes that $\gamma^{i-1} = w^p$. □

In Theorem 5-4, the columns w^p and w^q need not be distinct. If they are distinct, then y^i is a vertex of $I(P)^*$. Otherwise y^i is merely a boundary point of $I(P)^*$.

If y^i is a vertex of $I(P)^*$ then Theorem 5-4 gives us two inequalities

$$\begin{aligned} \sigma^{i-1} \cdot (x-x^i) &\leq 0 \\ \sigma^{i+1} \cdot (x-x^i) &\leq 0 \end{aligned} \tag{5-28}$$

in the system $A'x \leq b'$ defining $H(I(P))$. In (5-28), $\sigma^{i\pm1} = (\bar{A})^{-1} \gamma^{i\pm1}$ and $x^i = A^{-1} y^i$.

We now consider the following question: given a vertex y^i of $I(P)^*$, how do we generate neighboring vertices, if any, of y^i?

Let y^i be a vertex of $I(P)^*$. We assume that $z^i = b-y^i$ has been used to label two columns of W as before. We show how to generate slack vectors $z^{i-2} = b-y^{i-2}$ and $z^{i+2} = b-y^{i+2}$ of neighboring vertices y^{i-2} and y^{i+2} respectively.

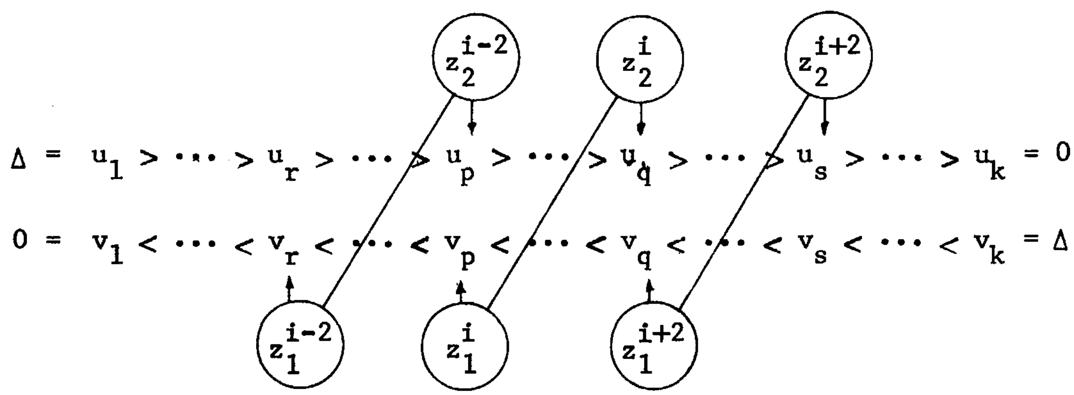
We refer to the previous labeling of W. If $v_p > 0$ (that is, $p > 1$) then let

$$z^{i-2} = z^i - \left\lfloor \frac{z_1^i}{v_p} \right\rfloor Tw^p. \tag{5-29}$$

Otherwise, disregard z^{i-2}. If $u_q > 0$ (that is, $q < k$) then let

$$z^{i+2} = z^i + \left\lfloor \frac{z_2^i}{u_q} \right\rfloor Tw^q. \tag{5-30}$$

Otherwise, disregard z^{i+2}. Using z^{i-2} and z^{i+2}, we label two additional columns of W as shown below:



v_r is the largest element in the lower row of W which is less than or equal to z_1^{i-2}. u_s is the largest element in the upper row of W which is less than or equal to z_2^{i+2}.

<u>Lemma 5-9</u>

y^{i-2} and y^{i+2} are vertices of $I(P)^*$.

Proof

We prove that y^{i+2} is a vertex of $I(P)^*$ by showing that w^q and w^s are two distinct normal vectors of $I(P)^*$ at y^{i+2}. We have

$$y^{i+2} = y^i - \left\lfloor \frac{z_2{}^i}{u_q} \right\rfloor Tw^q.$$

Clearly $y^{i+2} \leq b$, so that $y^{i+2} \in I(P)^*$. Since $w^q \cdot Tw^q = 0$, it follows that $w^q \cdot (y-y^{i+2}) = w^q \cdot (y-y^i) \leq 0$ for all $y \in I(P)^*$. Also, since $\left\lfloor \frac{z_2{}^i}{u_q} \right\rfloor \geq 1$, $y^{i+2} + Tw^q \in I(P)^*$. Thus w^q is a normal vector of $I(P)^*$ at y^{i+2}. By Theorem 5-4, w^s is a normal vector of $I(P)^*$ at y^{i+2}. Since $z_2{}^{i+2} < u_q$ we have $w^s \neq w^q$. Therefore y^{i+2} is a vertex of $I(P)^*$.

A similar argument reveals that y^{i-2} is a vertex of $I(P)^*$. □

What we have described may be viewed as a second level two-dimensional division process in which the slack vectors of known vertices are the dividends, atoms of $(M(\bar{A})^+, \leq)$ are divisors, and slack vectors of new vertices are the remainders.

In order to initiate the process we must have an initial vertex. This is easily obtained. In Section 5.1 we saw that $I(\pi_1) \cap I(P)$ and $I(\pi_2) \cap I(P)$ are sets of boundary points of $I(P)$, where π_1 and π_2 are boundary lines of half-planes defining P.

Furthermore we saw that $I(\pi_1) \cap I(P)$ is the set of integer points expressible as $x^1 - k_1 T a^1$, where x^1 is an integer point in $I(\pi_1)$ computed using the Euclidean algorithm and k_1 is an integer such that,

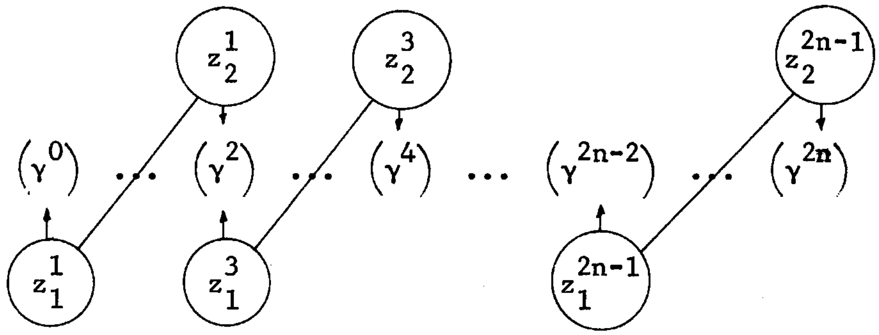
$$k_1 \leq \left\lfloor \frac{b_2 - a^2 \cdot x^1}{\Delta} \right\rfloor \quad .$$

We obtained a similar characterization of $I(\pi_2) \cap I(P)$.

If we take $x^1 - k_1 T a^1$, where $k_1 = \left\lfloor \frac{b_2 - a^2 \cdot x^1}{\Delta} \right\rfloor$, as being x^1, then we claim that x^1 is a vertex of $I(P)$. Consider $y^1 = Ax^1$. If we compute $z^1 = b - y^1$ we find that $z_1^1 = 0$ and $z_2^1 < \Delta$. It follows from Theorem 5-4 that w^p and w^q are normal vectors of $I(P)^*$ at y^1, where $p = 1$ and $q > 1$. Thus y^1 is a vertex of $I(P)^*$.

Similarly, we can find an initial vertex drawn from $I(\pi_2) \cap I(P)$.

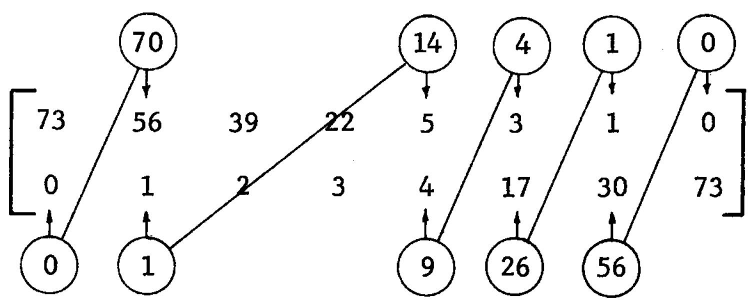
Using the slack vector z^1 of the initial vertex y^1 of $I(P)^*$, we label two columns of the matrix W as we did for Theorem 5-4. The circled slack vector z^1 then points to the normal vectors γ^0 and γ^2 of $I(P)^*$ at y^1. The division indicated by equation (5-30) is then performed on the labeled matrix W to obtain slack vector z^3, which in turn is used to label a third column of W. The circled slack vector z^3 then points to the normal vectors γ^2 and γ^4 — and so on. The diagram below illustrates the labeled matrix W.



<u>Example 5-6</u> Suppose W is given by

$$W = \begin{bmatrix} 73 & 56 & 39 & 22 & 5 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 17 & 30 & 73 \end{bmatrix}$$

and suppose $z^1 = \begin{pmatrix} 0 \\ 70 \end{pmatrix}$. Then the labeled matrix W is shown below.



Sample divisions are:

$$\begin{pmatrix} 1 \\ 14 \end{pmatrix} = \begin{pmatrix} 0 \\ 70 \end{pmatrix} + \left\lfloor \frac{70}{56} \right\rfloor \begin{pmatrix} 1 \\ -56 \end{pmatrix}$$

$$\begin{pmatrix} 9 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 14 \end{pmatrix} + \left\lfloor \frac{14}{5} \right\rfloor \begin{pmatrix} 4 \\ -5 \end{pmatrix} .$$

Notice that Equation (5-29) enables us to check our division by working back from right to left. □

The process terminates when a vertex y^{2n-1} is found such that $z_2^{2n-1} = 0$, which must occur after at most k-1 divisions have been performed, where k is the number of columns of W. If the division process is executed from right to left, beginning with a slack vector z^{-1}, then the process terminates when a vertex y^{-2n+1} is found such that $z_1^{-2n+1} = 0$.

In the final labeled matrix W, each circled slack vector z^i points to the two normal vectors γ^{i-1} and γ^{i+1} of $I(P)^*$ at vertex y^i. Each normal vector γ^i is pointed to by the one or two circled slack vectors representing vertices of $I(P)^*$ at which γ^i is a normal vector.

The final result of all this is an alternating sequence

$$\begin{array}{ccccccccc} \gamma^0 & & \gamma^2 & & \gamma^4 & \cdots & \gamma^{2n} \\ & y^1 & & y^3 & \cdots & y^{2n-1} & \end{array}$$

of normal vectors and vertices of $I(P)^*$, which may be transformed into the alternating sequence

$$\begin{array}{ccccccccc} \sigma^0 & & \sigma^2 & & \sigma^4 & \cdots & x^{2n} \\ & x^1 & & x^3 & \cdots & x^{2n-1} & \end{array}$$

of normal vectors and vertices of I(P), where $\sigma^{i\pm 1}=(\bar{A})^{-1}\gamma^{i\pm 1}$ and $x^i = A^{-1}y^i$, for i = 1, 3, ..., 2n-1. We then write the following system of inequalities,

$$
\begin{aligned}
&\sigma^0 \cdot (x-x^1) \le 0 \\
&\sigma^2 \cdot (x-x^1) \le 0 \\
&\sigma^4 \cdot (x-x^3) \le 0 \\
&\quad\vdots \\
&\sigma^{2n} \cdot (x-x^{2n-1}) \le 0.
\end{aligned}
\tag{5-31}
$$

Since $z_1^1 = 0$ and $z_2^{2n-1} = 0$, it follows from Theorem 5-4 that $\gamma^0 = \begin{pmatrix}\Delta\\0\end{pmatrix}$ and $\gamma^{2n} = \begin{pmatrix}0\\\Delta\end{pmatrix}$. Thus $\sigma^0 = (\bar{A})^{-1}\gamma^0 = a^1$ and $\sigma^0 \cdot x^1 = \frac{1}{\Delta}\ \gamma^0 \cdot y^1 = y_1^1 = b_1$, so that the first inequality in (5-31) is $a^1 \cdot x \le b_1$. Similarly $\sigma^{2n} = (\bar{A})^{-1}\gamma^{2n} = a^2$ and $\sigma^{2n} \cdot x^{2n-1} = \frac{1}{\Delta}\ \gamma^{2n} \cdot y^{2n-1} = y_2^{2n-1} = b_2$, so that the last inequality in (5-31) is $a^2 \cdot x \le b_2$. These two inequalities are the two which define P.

If we let Q denote the polyhedron defined by the inequalties (5-31), then it is easy to see that Q = H(I(P)). Since each of the half-planes whose intersection is Q contains I(P), it follows that I(P) $\subseteq$ I(Q). Since the two half-planes whose intersection is P are included, it follows that I(Q) $\subseteq$ I(P). Thus I(Q) = I(P). Furthermore, by construction, every vertex of Q is an integer point. Thus Q is integral. From Lemma 1-1, Q = H(I(P)).

The following lemma gives an upper bound on the number of vertices of H(I(P)).

Lemma 5-10

$$H(I(P)) \text{ has at most } \Delta \text{ vertices.}$$

Proof

We have previously argued that matrix W has at most $\Delta + 1$ columns. Therefore, in labeling the columns of W, at most Δ vertices are generated. □

We conclude this chapter with two examples illustrating the integralization of (2,2)-corner polyhedra.

Example 5-7 Let P be defined by

$$\begin{bmatrix} 2 & 5 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 72 \\ 2 \end{bmatrix}.$$

Then

$$\bar{A} = \begin{bmatrix} 3 & 5 \\ -5 & 2 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 31 & 18 \\ 0 & 1 \end{bmatrix}$$

$$W = \begin{bmatrix} 31 & 18 & 5 & 2 & 1 & 0 \\ 0 & 1 & 2 & 7 & 19 & 31 \end{bmatrix}.$$

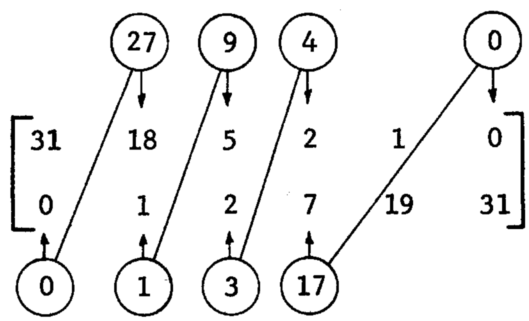
An initial vertex x^0 of I(P) is given by,

$$x^0 = 72\begin{pmatrix}-2\\1\end{pmatrix} - k\begin{pmatrix}5\\-2\end{pmatrix}$$

where

$$k = \left\lfloor \frac{2-\begin{pmatrix}-5\\3\end{pmatrix}\cdot 72\begin{pmatrix}-2\\1\end{pmatrix}}{31} \right\rfloor = -31$$

Thus $x^0 = \begin{pmatrix}11\\10\end{pmatrix}$, and $z^0 = \begin{pmatrix}0\\27\end{pmatrix}$. W is labeled as follows:



The resulting system of inequalities defining H(I(P)) is:

$$\begin{aligned}
\begin{pmatrix}2\\5\end{pmatrix} \cdot \left(x-\begin{pmatrix}11\\10\end{pmatrix}\right) &\le 0\\
\begin{pmatrix}1\\3\end{pmatrix} \cdot \left(x-\begin{pmatrix}11\\10\end{pmatrix}\right) &\le 0\\
\begin{pmatrix}0\\1\end{pmatrix} \cdot \left(x-\begin{pmatrix}8\\11\end{pmatrix}\right) &\le 0\\
\begin{pmatrix}-1\\1\end{pmatrix} \cdot \left(x-\begin{pmatrix}7\\11\end{pmatrix}\right) &\le 0\\
\begin{pmatrix}-5\\3\end{pmatrix} \cdot \left(x-\begin{pmatrix}5\\9\end{pmatrix}\right) &\le 0
\end{aligned}
\qquad \text{or} \qquad
\begin{bmatrix} 2 & 5\\ 1 & 3\\ 0 & 1\\ -1 & 1\\ -5 & 3 \end{bmatrix}
\begin{bmatrix} x_1\\ x_2 \end{bmatrix} \le
\begin{bmatrix} 72\\ 41\\ 11\\ 4\\ 2 \end{bmatrix} .$$

$\square$

<u>Example 5-8</u> Let P be defined by

$$\begin{bmatrix} 488 & 216 \\ 46 & 103 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 279 \\ -317 \end{bmatrix}$$

Using the Euclidean algorithm we find that gcd (488, 216) = 8 and gcd (46, 103) = 1. Since $8 \nmid 279$ the first inequality does not define an integral half-plane. The second one does. We replace the first by $61\ x_1 + 27\ x_2 \le \left\lfloor \frac{279}{8} \right\rfloor$ and obtain a new polyhedron P_1 defined by

$$\begin{bmatrix} 61 & 27 \\ 46 & 103 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 34 \\ -317 \end{bmatrix}$$

Then

$$\bar{A} = \begin{bmatrix} 103 & -46 \\ -27 & 61 \end{bmatrix} .$$

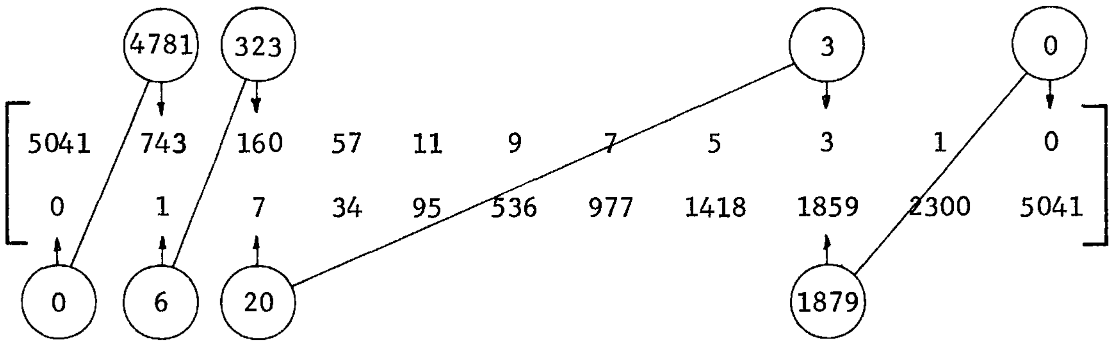
Matrix W has been computed in Example 5-5. An initial vertex x^0 of $I(P_1)$ is given by,

$$x^0 = 34 \begin{pmatrix} 4 \\ -9 \end{pmatrix} - k \begin{pmatrix} 27 \\ -61 \end{pmatrix}$$

where

$$k = \left\lfloor \frac{-317 - \begin{pmatrix} 46 \\ 103 \end{pmatrix} \cdot 34 \begin{pmatrix} 4 \\ -9 \end{pmatrix}}{5041} \right\rfloor = 4$$

Thus $x^0 = \begin{pmatrix} 28 \\ -62 \end{pmatrix}$ and $z^0 = \begin{pmatrix} 0 \\ 4781 \end{pmatrix}$. W is labeled as follows:



The resulting system of inequalities defining $H(I(P_1)$ is:

$$\begin{pmatrix} 61 \\ 27 \end{pmatrix} \cdot \left(x - \begin{pmatrix} 28 \\ -62 \end{pmatrix}\right) \le 0$$

$$\begin{pmatrix} 9 \\ 4 \end{pmatrix} \cdot \left(x - \begin{pmatrix} 28 \\ -62 \end{pmatrix}\right) \le 0$$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \left(x - \begin{pmatrix} 4 \\ -8 \end{pmatrix}\right) \le 0 \qquad \text{or} \qquad \begin{bmatrix} 61 & 27 \\ 9 & 4 \\ 2 & 1 \\ 17 & 38 \\ 46 & 103 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 34 \\ 4 \\ 0 \\ -118 \\ -317 \end{bmatrix} .$$

$$\begin{pmatrix} 17 \\ 38 \end{pmatrix} \cdot \left(x - \begin{pmatrix} 2 \\ -4 \end{pmatrix}\right) \le 0$$

$$\begin{pmatrix} 46 \\ 103 \end{pmatrix} \cdot \left(x - \begin{pmatrix} -36 \\ 13 \end{pmatrix}\right) \le 0$$

$\square$

CHAPTER 6

INTEGRALIZATION OF (2,2)-POLYHEDRA

Let P be a (2,2)-polyhedron defined by $Ax \le b$, where A is an integer $m \times 2$ matrix of rank 2, and b is an integer m-vector. In this chapter we apply the integralization method of Chapter 5 to the problem of integralizing P.

We begin by reducing P to an intersection of integral half-planes by replacing each inequality $a^i \cdot x \le b_i$ in $Ax \le b$ by $\frac{1}{d_i} a^i \cdot x \le \left\lfloor \frac{b_i}{d_i} \right\rfloor$ where $d_i = \gcd(a_{i1}, a_{i2})$. If we let τ_i denote the half-plane defined by $a^i \cdot x \le b_i$ and τ_i' denote the half-plane defined by $\frac{1}{d_i} a^i \cdot x \le \left\lfloor \frac{b_i}{d_i} \right\rfloor$, then $I(\tau_i') = I(\tau_i)$ for $i = 1, 2, \ldots, m$. Since $I(P) = \bigcap_{i=1}^{m} I(\tau_i) = \bigcap_{i=1}^{m} I(\tau_i')$, P may be replaced by $\bigcap_{i=1}^{m} \tau_i'$ without changing I(P).

In the following we assume that this first integralization step has already been performed on P. That is, we assume that each row $[a_{i1} \ \ a_{i2}]$ of A has the property $\gcd(a_{i1}, a_{i2}) = 1$.

Our plan is to integralize P by generating all vertices and normal vectors of I(P). Boundary points, vertices and normal vectors of I(P) are defined just as they were for (2,2)-corner polyhedra in the previous chapter. We accomplish this by generating boundary points and normal vectors for a sequence of (2,2)-corner polyhedra which we shall call supporting corners of I(P).

A (2,2)-corner polyhedron Q is a <u>supporting corner of I(P)</u> iff $I(P) \subseteq I(Q)$ and I(Q) and I(P) have one or more vertices in common.

In the following we show how an initial supporting corner of I(P) is found. We then show how supporting corners of I(P) are used to generate vertices, normal vectors and other supporting corners of I(P). We conclude by showing that the process terminates after a finite number of steps, resulting in a system of inequalities $A'x \leq b'$ defining P'.

6.1 Finding an Initial Supporting Corner

Well-known methods associated with linear programming (see, for example, Dantzig [5]) may be used to find a vertex v of P. Associated with vertex v is a two-inequality subsystem $Cx \leq d$ of the system $Ax \leq b$ defining P, where $v = C^{-1}d$. $Cx \leq d$ defines a (2,2)-corner polyhedron Q such that $I(P) \subseteq I(Q)$.

Applying the methods of Chapter 5 to Q, we obtain the sequence

$$\begin{array}{ccccccccc} \sigma^0 & & \sigma^2 & & \sigma^4 & \cdots & \sigma^{2n} \\ & x^1 & & x^3 & & \cdots & & x^{2n-1} \end{array}$$

of normal vectors and vertices of I(Q). Suppose $x^i \in I(P)$ for some i, where i = 1, 3, ..., 2n-1. Since $\sigma^{i\pm 1}\cdot(x-x^i) \leq 0$ for all $x \in I(Q)$, it follows from the fact that $I(P) \subseteq I(Q)$ that $\sigma^{i\pm 1}\cdot(x-x^i) \leq 0$ for all $x \in I(P)$. Thus x^i is a common vertex of both I(Q) and I(P), and Q is a supporting corner of I(P).

Now suppose $x^i \notin I(P)$ for i = 1, 3, ..., 2n-1. In this case, P either does or does not intersect the boundary of Q' = H(I(Q)).

Suppose P does not intersect the boundary of Q'. Then since the vertex v is in P and P is convex, it follows that P lies wholly within Q-Q' and thus I(P) = φ and P' = φ, where φ denotes the empty set.

Suppose on the other hand that P does intersect the boundary of Q'. Let e^i, for i = 1, 3, ..., 2n-3, denote the edge

$$\{x \in R^2 \mid x = x^i + \lambda(x^{i+2}-x^i),\ 0 \leq \lambda \leq 1\}$$

of Q'. We argue that the intersection of P with the boundary of Q' is containing in exactly one edge e^i of Q', as illustrated in Fig. 6-1.

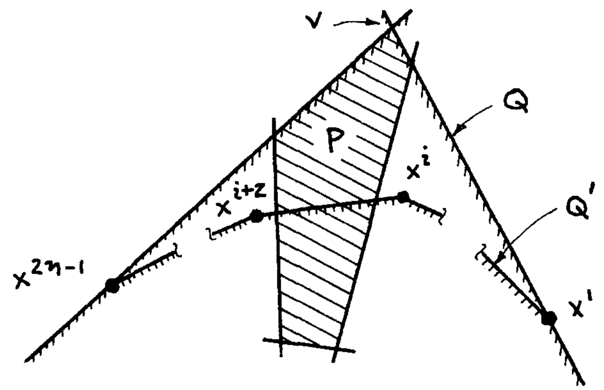


Fig. 6-1

First, we see that neither the half-line edge

$$\{x \in R^2 \mid x = x^1 + \lambda(x^1 - v),\ \lambda \geq 0\}$$

nor the half-line edge

$$\{x \in R^2 \mid x = x^{2n-1} + \lambda(x^{2n-1} - v),\ \lambda \geq 0\}$$

of Q' contains any points in P. Otherwise, since $v \in P$, the convexity of P implies that $x^1 \in I(P)$ or $x^{2n-1} \in I(P)$, contrary to assumption. Next we see that if two edges e_i and e_j of Q', where $i < j$, both contain points in P, then since $v \in P$, the convexity of P implies that x^{i+2}, x^{i+4}, ..., $x^j \in I(P)$, again contrary to assumption. Thus the intersection of P with the boundary of Q' is contained in exactly one edge e^i of Q'. We form a new (2,2)-polyhedron P^1 by adding the inequality $\sigma^{i+1} \cdot (x - x^i) \leq 0$ to the system $Ax \leq b$ and deleting the subsystem $Cx \leq d$. Since $P-P^1$ lies wholly within $Q-Q'$ and thus contains no integer points, we see that $I(P^1) = I(P)$. Thus we have reduced the number of inequalities defining P by one, without changing I(P).

Computationally, these observations are implemented as follows. For each vertex x^i of I(Q), $i = 1, 3, \ldots, 2n-1$, we compute the slack vector $z^i = b-Ax^i$. If $z^i \geq 0$ for some i, then $x^i \in I(P)$ and, as we have seen, x^i is a common vertex of both I(Q) and I(P). Thus Q is a supporting corner of I(P).

Otherwise $z^i \ngeq 0$ and $x^i \notin I(P)$ for $i = 1, 3, \ldots, 2n-1$. In this case, for each i, $i = 1, 3, \ldots, 2n-3$, we form the following system of $m + 2$ inequalities in λ, given below in vector form as,

$$\begin{aligned} z^i + \lambda(z^{i+2}-z^i) &\geq 0 \\ 0 \leq \lambda &\leq 1. \end{aligned} \tag{6-1}$$

(6-1) has a solution λ iff P intersects the boundary of Q' at edge e^i. As we have seen, two cases are possible. In the first, (6-1) has no solution λ, for $i = 1, 3, \ldots, 2n-3$. In this case $I(P) = \varphi$. In the second case, (6-1) has a solution λ for exactly one i among $1, 3, \ldots, 2n-3$. We then form a new (2,2)-polyhedron P^1 by adding the inequality $\sigma^{i+1}\cdot(x-x^i) \leq 0$ to $Ax \leq b$ and deleting the subsystem $Cx \leq d$.

If we have not found a supporting corner of I(P) or learned that $I(P) = \varphi$, we may apply the entire procedure described above to P^1. In general, during the k^{th} application of this procedure we either: (a) discover a supporting corner of I(P), or (b) find that $I(P) = \varphi$, or (c) obtain a (2,2)-polyhedron P^k defined by m-k inequalities such that $I(P^k) = I(P)$.

We terminate this procedure when either (a) or (b) results from a given pass. If neither (a) nor (b) have resulted from the first m-2 passes, then we are left with a (2,2)-corner polyhedron P^{m-2} such that $I(P^{m-2}) = I(P)$. By definition, P^{m-2} is a supporting corner of I(P).

We assume now and for the remainder of this chapter that $I(P) \neq \varphi$. Thus a supporting corner of I(P) is found on the k^{th} pass, where $1 \leq k \leq m-2$. In the following, for convenience, we refer to the resulting (2,2)-polyhedron P^{k-1} as P and to its defining system of inequalities as $Ax \leq b$.

6.2 Generation of Normal Vectors and Vertices

Let Q^k be a supporting corner of I(P) and let x^i be a common vertex of both $I(Q^k)$ and I(P). We refer to the pair (Q^k, x^i) as a <u>state</u> of the integralization process. Q^k is defined by two inequalities $C^k x \leq d^k$. We assume they are ordered such that $\det C^k > 0$. We shall refer to the inequality toward whose boundary line we are moving as the <u>objective inequality</u>. The objective inequality associated with Q^k is drawn from the system $Ax \leq b$ defining P. We shall denote the objective inequality associated with Q^k by $a^{j_k} \cdot x \leq b_{j_k}$, the j_k^{th} inequality in $Ax \leq b$.

We proceed from state to state by means of a basic iterative step. A state which results from one application of this iterative step to (Q^k, x^i), where the second (first) inequality of $C^k x \leq d^k$ is the objective inequality, is A + (-) successor state of (Q^k, x^i). The $\pm$ successor of (Q^k, x^i) takes one of the three forms illustrated in the diagram below.

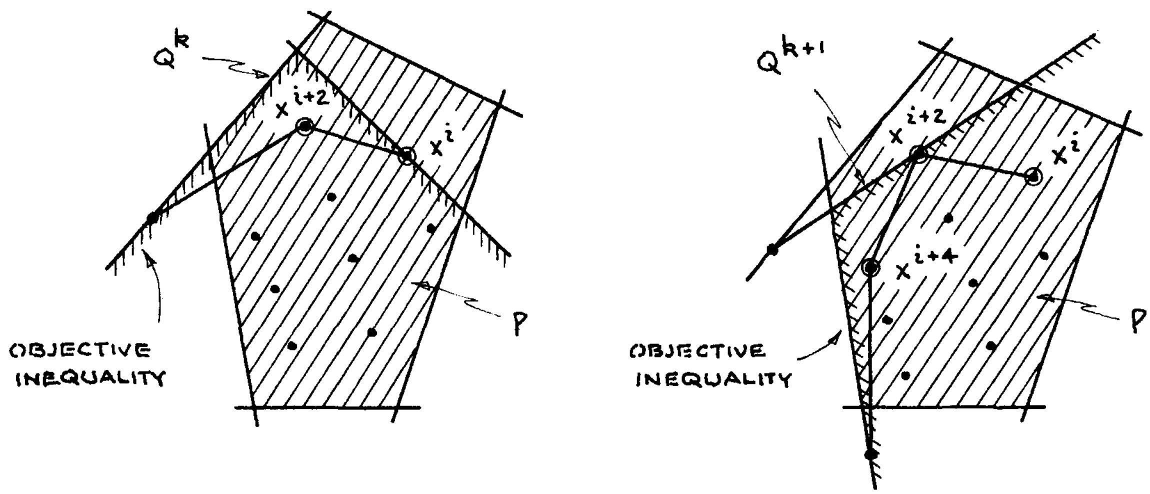
$$\begin{tikzcd}
 & (Q^k, x^i) \arrow[dl] \arrow[d] \arrow[dr] & \\
(Q^k, x^{i\pm 2}) & (Q^{k\pm 1}, x^{i\pm 2}) & (Q^{k\pm 1}, x^i)
\end{tikzcd}$$

The $\pm$ successor of (Q^k, x^i) is characterized by a new vertex, a new supporting corner, or both.

<u>Example 6-1</u> Fig. 6-2(a) and (b) illustrate a succession of states: $(Q^k, x^i) \rightarrow (Q^k, x^{i+2}) \rightarrow (Q^{k+1}, x^{i+2}) \rightarrow (Q^{k+1}, x^{i+4})$. □

A state (Q^0, x^0) used to start the process is an <u>initial state</u>. Q^0 is defined by two inequalities $C^0 x \leq d^0$ both drawn from $Ax \leq b$.

Thus either inequality may be designated as the objective inequality $a^{j_0} \cdot x \leq b_{j_0}$. We saw how to find an initial state in the previous section.



(a) (b)

Fig. 6-2

A state which has no + successor or no - successor is a <u>terminal state</u>.

We now describe the basic iterative step for determining the $\pm$ successor† of (Q^k, x^i), if one exists.

† For + successor, all instances of $\pm(\mp)$ should be read as $+(-)$. The opposite interpretation holds for - successor.

(1) Using the (2,2)-corner polyhedron integralization method of Chapter 5, we compute the normal vector $\sigma^{i\underline{+}1}$ of $I(Q^k)$ at x^i.

(2) We let $\delta x^i = \mp T\sigma^{i\underline{+}1}$. We compute

$$\delta y^i = A\ \delta x^i$$

and compare δy^i with the slack vector z^i, where

$$z^i = b - Ax^i = b - y^i.$$

(3) If all components of δy^i are non-positive then $x^i + n\ \delta x^i \in I(P)$ for all integers $n \geq 0$. $\sigma^{i\underline{+}1}$ is a normal vector of $I(P)$ at x^i. We adjoin the inequality $\sigma^{i\underline{+}1}.(x-x^i) \leq 0$ to an evolving system of inequalities defining P'. The state (Q^k, x^i) is a terminal state and has no $\underline{+}$ successor.

(4) If one or more components of δy^i are positive then we compute the non-negative rational numbers

$$\lambda j = \frac{z_j^i}{\delta y_j^i}$$

for all subscripts j such that $\delta y_j^i > 0$, and among these we identify a subscript $j_{k\underline{+}1}$ such that

$$\lambda j_{k\underline{+}1} \leq \lambda j$$

for all j such that $\delta y_j^{\,i} > 0$.† We distinquish three possible cases corresponding to the three possible forms of the $\pm$ successor of (Q^k, x^i): (a) $j_{k\pm1} = j_k$, (b) $j_{k\pm1} \neq j_k$ and $\lambda j_{k\pm1} \geq 1$, (c) $j_{k\pm1} \neq j_k$ and $0 \leq \lambda j_{k\pm1} < 1$.

(5a) $(j_{k\pm1} = j_k)$ In this case $\lambda j_k \geq 1$. Since $x^i + \delta x^i \in I(P)$, $\sigma^{i\pm1}$ is a normal vector of I(P) at x^i. We adjoin the inequality $\sigma^{i\pm1}\cdot(x-x^i) \leq 0$ to an evolving system of inequalities defining P'. We let

$$x^{i\pm2} = x^i + \lfloor \lambda j_k \rfloor \, \delta x^i.$$

$x^{i\pm2}$ is a vertex of both $I(Q^k)$ and I(P). We take $(Q^k, x^{i\pm2})$ as the $\pm$ successor of (Q^k, x^i).

(5b) $(j_{k\pm1} \neq j_k$ and $\lambda j_{k\pm1} \geq 1)$ In this case, since $x^i + \delta x^i \in I(P)$, $\sigma^{i\pm1}$ is a normal vector of I(P) at x^i. We adjoin the inequality $\sigma^{i\pm1}\cdot(x-x^i) \leq 0$ to an evolving system of inequalities defining P'. We let

$$x^{i\pm2} = x^i + \lfloor \lambda j_{k\pm1} \rfloor \, \delta x^i.$$

† The following somewhat arbitrary rule insures the determinism of this step: If there is more than one subscript j such that λj is smallest, then choose j_k if it is included — otherwise choose the smallest subscript. (Recall that j_k identifies the objective inequality associated with Q^k.)

In addition, we let $Q^{k\pm1}$ be the (2,2)-corner polyhedron defined by $\sigma^{i\pm1}\cdot(x-x^i) \leq 0$ and the new objective inequality $a^{j_{k\pm1}}\cdot x \leq b_{j_{k\pm1}}$ drawn from $Ax \leq b$. We see that $I(P) \subseteq I(Q^{k\pm1})$ and that $x^{i\pm2}$ is a vertex of both $I(Q^{k\pm1})$ and $I(P)$. Thus $Q^{k\pm1}$ is a supporting corner of $I(P)$. We take $(Q^{k\pm1}, x^{i\pm2})$ as the $\pm$ successor of (Q^k, x^i).

(5c) ($j_{k\pm1} \neq j_k$ and $0 \leq \lambda j_{k\pm1} < 1$) In this case, since $x^i + \delta x^i \notin I(P)$, no new vertex is obtained. We let $Q^{k\pm1}$ be the (2,2)-corner polyhedron defined by $\sigma^{i\pm1}\cdot(x-x^i) \leq 0$ and the new objective inequality $a^{j_{k\pm1}}\cdot x \leq b_{j_{k\pm1}}$ drawn from $Ax \leq b$. We see that $I(P) \subseteq I(Q^{k\pm1})$ and that x^i is a vertex of both $I(Q^{k\pm1})$ and $I(P)$. We take $(Q^{k\pm1}, x^i)$ as the $\pm$ successor of (Q^k, x^i).

This completes our description of the iterative step.

The integralization process proceeds as follows. Starting with an initial state (Q^0, x^0), a sequence S^+ of + successors is generated until a state (Q^k, x^{2n}) is obtained such that:

(a) $n \geq 2$ and $x^{2n} = x^0$

or

(b) (Q^k, x^{2n}) is a terminal state.

If (Q^k, x^{2n}) is a terminal state then starting with the initial state (Q^0, x^0), a sequence S^- of - successors is generated until a terminal state $(Q^{-k'}, x^{-2n'})$ is obtained.

In case (a), the sequence S^+ of states identifies n distinct vertices $x^0, x^2, \ldots, x^{2n-2}$ of I(P) and returns to the initial vertex x^0. If $n > 2$ then the generated inequalities $\sigma^{i+1}\cdot(x-x^i) \leq 0$, for $i = 0, 2, \ldots, 2n-2$, define a bounded (2,2)-polyhedron $\bar{P}$ having n vertices. $\bar{P}$ is contained in P so that $I(\bar{P}) \subseteq I(P)$. Also, since each of these inequalities is satisfied for all $x \in I(P)$, we have $I(P) \subseteq I(\bar{P})$. Furthermore, each vertex of $\bar{P}$ is an integer point, so that $\bar{P}$ is integral. By Lemma 1-1, $\bar{P} = P'$. If $n = 2$ then P' is the line segment having x^0 and x^2 as endpoints. Only two inequalities are generated. They define the line containing P'.

In case (b), the sequence S^+ of states identifies n+1 distinct vertices $x^0, x^2, \ldots, x^{2n}$ of I(P). Beyond vertex x^{2n} we discover an infinite number of boundary points of I(P). It follows that P is unbounded. The sequence S^- of states then identifies n'+1 distinct vertices $x^0, x^{-2}, \ldots, x^{-2n'}$ of I(P). Beyond vertex $x^{-2n'}$ we discover an infinite number of boundary points of I(P). The generated inequalities, $\sigma^{i+1}\cdot(x-x^i) \leq 0$ for $i = 0, 2, \ldots, 2n$ and $\sigma^{i-1}\cdot(x-x^i) \leq 0$ for $i = 0, -2, \ldots, -2n'$, define an unbounded (2,2)-polyhedron $\bar{P}$ having n + n' + 1 vertices. An argument similar to the one given for case (a) establishes that $\bar{P} = P'$.

In the following we show that the integralization process terminates after a finite number of states have been generated.

Consider the sequence S^+ of + successors of an initial state (Q^0, x^0). We divide S^+ into subsequences $S_0^+, S_1^+, S_2^+, \ldots$, where

S_k^+, for $k = 0, 1, 2, \ldots,$ consists of the consecutive block of states having the same supporting corner Q^k as first component. Each supporting corner Q^k is defined by a system of two inequalities $C^k x \leq d^k$. According to Lemma 5-10, $(Q^k)'$ has at most $\Delta_k = |\det C^k|$ vertices. It follows that the subsequence S_k^+ contains at most Δ_k states.

Now consider the subsequences S_0^+, S_1^+, S_2^+, $\ldots$. Associated with each subsequence is a supporting corner Q^k. Associated with corner Q^k is an objective inequality drawn from the system of m inequalities $Ax \leq b$ defining P. This inequality is identified by the subscript j_k. Consider the sequence $j_0, j_1, j_2, \ldots$. There are two cases to consider:

(i) The sequence contains a j_k such that $j_k = j_i$ for some $i < k$. More simply, the sequence contains a repeat. We assume that j_k is the first repeat in the sequence. In this case, it can be shown that x^0 is a vertex of $I(Q^k)$. It follows that, for some state (Q^k, x^{2n}) in S_k^+, $x^{2n} = x^0$. If $n = 0$ then we have the degenerate case in which I(P) consists of the single integer point x^0. Otherwise, $n \geq 2$ (the case $n = 1$ is not possible since x^2 is necessarily distinct from x^0) and we have a state, as in case (a) above, at which S^+ terminates. Thus S_k^+ is the final subsequence of S^+. Since each member of the sequence $j_0, j_1, j_2, \ldots, j_k$ has one of m possible values and j_k is the first repeat, it follows that $k \leq m$.

(ii) The sequence $j_0, j_1, j_2, \dots$ contains a j_k which identifies an objective inequality drawn from $Ax \leq b$ whose boundary line contains a half-line edge of P. In this case, some state (Q^k, x^{2n}) in S_k^+ is a terminal state, as in case (b) above. Thus S_k^+ is the final subsequence of S^+. The sequence $j_0, j_1, \dots, j_k$ contains no repeats, otherwise case (i) would apply. It follows that $k < m$.

In either case, the number of subsequences $S_0^+, S_1^+, \dots, S_k^+$ in S^+ is at most $m + 1$. Thus S^+ is a finite sequence of states. A similar argument can be made for a sequence S^- of - successors. It follows that the integralization process terminates after a finite number of states have been generated.

We conclude this chapter with an example illustrating the integralization process for a (2,2)-polyhedron.

<u>Example 6-2</u> Let P be defined by

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -18 & 8 \\ -9 & 12 \\ 4 & 5 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 21 \\ 56 \\ 45 \\ 68 \end{bmatrix}$$

We reduce P to the following intersection of integral half-planes,

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -9 & 4 \\ -3 & 4 \\ 4 & 5 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ 10 \\ 18 \\ 45 \\ 68 \end{bmatrix} .$$

An initial state (Q^0, x^0) is found to be,

$$Q^0: \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

We generate a sequence of + successors of (Q^0, x^0). The steps involved are displayed by giving a tabulation of the resulting z^i and δy^i vectors, followed by the sequence of supporting corners and labeled W matrices. In the tabulation, we circle the component of z^i for which λ_j is smallest. Inequalities defining P' are enclosed in boxes as they are generated.

		z^0	δy^0	z^2	δy^2	z^4	δy^4	z^4	δy^4	z^6	δy^6	z^8	δy^8
-1	0	0	-1	8	0	8	1	8	1	7	1	5	5
0	-1	0	0	0	-1	1	-3	1	-2	3	-1	5	-4
-9	4	10	-9	82	4	78	21	78	17	61	13	35	61
-3	4	18	-3	42	4	38	15	38	11	27	7	13	31
4	5	45	4	13	5	8	11	8	6	2	1	0	0
8	3	68	8	4	3	1	1	1	-2	3	-5	13	-28

z^8	δy^8	z^{10}	δy^{10}	z^{12}	δy^{12}	z^{12}	δy^{12}	z^{14}	δy^{14}	z^{16}
5	2	3	1	2	4	2	1	0	0	0
5	-1	6	0	6	3	6	2	2	1	0
35	22	13	9	4	24	4	1	2	-4	10
13	10	3	3	0	0	0	-5	10	-4	18
0	-3	3	-4	7	-31	7	-14	35	-5	45
13	-13	26	-8	34	-41	34	-14	62	-3	68

$$Q^0: \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad (\bar{A})^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$W = \begin{array}{cc} & 0 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \hspace{-1.2em}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 & \end{array}$$

$$\begin{array}{cccc} x^0 & \gamma^1 & \sigma^1 & \delta x^0 \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array} \qquad \boxed{-x_2 \le 0}$$

$$Q^1: \begin{bmatrix} 0 & -1 \\ 8 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 0 \\ 68 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 3 & -8 \\ 1 & 0 \end{bmatrix}, \quad (\bar{A})^{-1} = \frac{1}{8} \begin{bmatrix} 0 & 8 \\ -1 & 3 \end{bmatrix}$$

$$W = \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{array}{cccc} & 4 & 1 & \\ \left[\begin{matrix} 8 \\ 0 \end{matrix}\right. & \begin{matrix} 3 \\ 1 \end{matrix} & \begin{matrix} 1 \\ 3 \end{matrix} & \left.\begin{matrix} 0 \\ 8 \end{matrix}\right] \\ 0 & 1 & & \end{array}$$

$$\begin{array}{cccc} x^2 & \gamma^3 & \sigma^3 & \delta x^2 \\ \begin{bmatrix} 8 \\ 0 \end{bmatrix} & \begin{bmatrix} 3 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{array} \quad \boxed{x_1 \le 8}$$

$$\begin{array}{cccc} x^4 & \gamma^5 & \sigma^5 & \delta x^4 \\ \begin{bmatrix} 8 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 3 \end{bmatrix} & \begin{bmatrix} 3 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 3 \end{bmatrix} \end{array}$$

$$Q^2: \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 25 \\ 45 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 5 & -4 \\ -1 & 3 \end{bmatrix}, \quad (\bar{A})^{-1} = \frac{1}{11} \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$$

$$W = \begin{array}{cccc} & 8 & 2 & 0 \\ \left[\begin{matrix} 11 \\ 0 \end{matrix}\right. & \begin{matrix} 6 \\ 1 \end{matrix} & \begin{matrix} 1 \\ 2 \end{matrix} & \left.\begin{matrix} 0 \\ 11 \end{matrix}\right] \\ 0 & 1 & 5 & \end{array}$$

$$\begin{array}{cccc} x^4 & \gamma^5 & \sigma^5 & \delta x^4 \\ \begin{bmatrix} 8 \\ 1 \end{bmatrix} & \begin{bmatrix} 6 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 2 \end{bmatrix} \end{array} \quad \boxed{2x_1 + x_2 \le 17}$$

$$\begin{array}{cccc} x^6 & \gamma^7 & \sigma^7 & \delta x^6 \\ \begin{bmatrix} 7 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{array} \quad \boxed{x_1 + x_2 \le 10}$$

$$\begin{array}{cccc} x^8 & \gamma^9 & \sigma^9 & \delta x^8 \\ \begin{bmatrix} 5 \\ 5 \end{bmatrix} & \begin{bmatrix} 0 \\ 11 \end{bmatrix} & \begin{bmatrix} 4 \\ 5 \end{bmatrix} & \begin{bmatrix} -5 \\ 4 \end{bmatrix} \end{array}$$

$$Q^3: \begin{bmatrix} 4 & 5 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 45 \\ 18 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 4 & 3 \\ -5 & 4 \end{bmatrix}, (\bar{A})^{-1} = \frac{1}{31} \begin{bmatrix} 4 & -3 \\ 5 & 4 \end{bmatrix}$$

$$W = \begin{array}{c} \begin{array}{cccccccc} & & & 13 & 3 & & & 0 \end{array} \\ \begin{bmatrix} 31 & 24 & 17 & 10 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 13 & 22 & 31 \end{bmatrix} \\ \begin{array}{cccccccc} 0 & & & 3 & 7 & & & \end{array} \end{array}$$

$$\begin{array}{cccc} x^8 & \gamma^9 & \sigma^9 & \delta x^8 \\ \begin{bmatrix} 5 \\ 5 \end{bmatrix} & \begin{bmatrix} 10 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} & \begin{bmatrix} -2 \\ 1 \end{bmatrix} \end{array} \quad \boxed{x_1 + 2x_2 \le 15}$$

$$\begin{array}{cccc} x^{10} & \gamma^{11} & \sigma^{11} & \delta x^{10} \\ \begin{bmatrix} 3 \\ 6 \end{bmatrix} & \begin{bmatrix} 3 \\ 4 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{array} \quad \boxed{x_2 \le 6}$$

$$\begin{array}{cccc} x^{12} & \gamma^{13} & \sigma^{13} & \delta x^{12} \\ \begin{bmatrix} 2 \\ 6 \end{bmatrix} & \begin{bmatrix} 0 \\ 31 \end{bmatrix} & \begin{bmatrix} -3 \\ 4 \end{bmatrix} & \begin{bmatrix} -4 \\ -3 \end{bmatrix} \end{array}$$

$$Q^4: \begin{bmatrix} -3 & 4 \\ -9 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 18 \\ 10 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 4 & 9 \\ -4 & -3 \end{bmatrix}, \quad (\bar{A})^{-1} = \frac{1}{24} \begin{bmatrix} -3 & -9 \\ 4 & 4 \end{bmatrix}$$

$$W = \begin{array}{c} \begin{array}{cccc} & & 4 & \end{array} \\ \begin{bmatrix} 24 & 5 & 1 & 0 \\ 0 & 1 & 5 & 24 \end{bmatrix} \\ \begin{array}{cccc} 0 & & & \end{array} \end{array}$$

$$\begin{array}{cccc} x^{12} & \gamma^{13} & \sigma^{13} & \delta x^{12} \\ \begin{bmatrix} 2 \\ 6 \end{bmatrix} & \begin{bmatrix} 1 \\ 5 \end{bmatrix} & \begin{bmatrix} -2 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ -2 \end{bmatrix} \end{array} \quad \boxed{-2x_1 + x_2 \le 2}$$

$$Q^5: \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad (\bar{A})^{-1} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

0 0

$$\begin{array}{cccc} x^{14} & \gamma^{15} & \sigma^{15} & \delta x^{14} \\ \begin{bmatrix} 0 \\ 2 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} -1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \end{bmatrix} \end{array} \quad \boxed{-x_1 \le 0}$$

$$\begin{array}{c} x^{16} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}$$

The sequence of + successors terminates with (Q^5, x^{16}) since $x^{16} = x^0$.

Fig. 6-3 illustrates P and H(I(P)). □

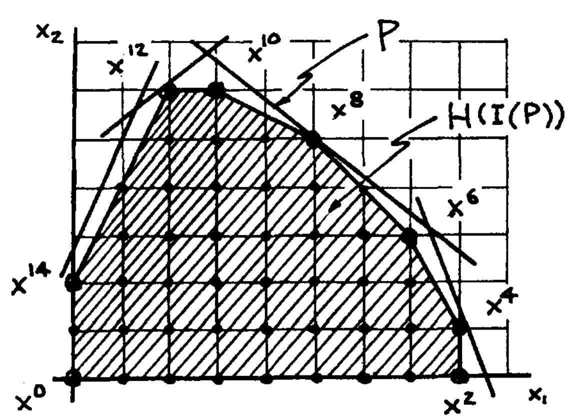


Fig. 6-3

contained in the non-negative orthant). Extending this algebraic characterization to include all totally integral polyhedra would be an interesting problem for future study.

We have answered the second of the two questions above for all (n,k)-polyhedra, where $k \leq 2$. We began in Chapter 4 by showing that an arbitrary (n,k)-polyhedron can be transformed into a (k,k)-polyhedron by a transformation which preserves integralization. This enables us to focus on the problem of integralizing (k,k)-polyhedra without loss of generality. We then described the solution to the simple case in which $k = 1$.

In Chapter 5 we developed our main integralization results for the case $k = 2$. We considered the problem of integralizing a (2,2)-corner polyhedron. Using a transformation which preserves the hull-forming operation, we showed that vectors normal to the surface of the sought after integral polyhedron are necessarily atoms of a certain partially ordered set. We introduced a two-dimensional generalization of the division theorem for integers, and showed how this is used to generate the finite set of all such atoms. We then described a second level division process involving these atoms which generates all vertices and normal vectors of the integral polyhdron, and thus its defining system of inequalities.

In Chapter 6 we considered the integralization problem for (2,2)-polyhedra in general. We showed how a sequence of supporting

(2,2)-corner polyhedra is generated and how the integralization method of Chapter 5 is applied to each, thereby producing all vertices and normal vectors of the sought after integral polyhedron, and its defining system of inequalities.

The integralization process developed in Chapters 5 and 6 is number theoretic in nature. It makes use of simple arithmetic operations without recourse to exhaustive enumerative methods or search procedures.

It had been our goal to generalize this integralization method to make it apply to (k,k)-polyhedra, where k is arbitrary. However, we encountered some difficulty in attempting to generalize to $k = 3$ which we now describe.

Let P be a (3,3)-corner polyhedron defined by $Ax \leq b$, where A is an integer 3×3 matrix with non-zero determinant and b is an integer 3-vector. We may integralize each of the three (3,1)-polyhedra (half-spaces) forming P. This gives us a (3,3)-corner polyhedron P^1 defined by a system $A^1x \leq b^1$ of three inequalities for which

$$\gcd\ [A^1_S] = \gcd\ [A^1_S \,\vdots\, b^1_S]$$

for all face 1-subsets S. We may then integralize each of the three (3,2)-polyhedra defined by two inequalities drawn from $A^1x \leq b^1$. Thus gives us a (3,3)-polyhedron P^2 defined by the system $A^2x \leq b^2$ of three or more inequalities for which

$$\gcd\,[A_S^2] = \gcd\,[A_S^2 \mid b_S^2]$$

for all face 2-subsets S. Thus far we have used one and two-dimensional integralization methods to obtain P^2. The remaining integralization step is three-dimensional in nature and this is where the difficulty arises.

To illustrate the difficulty in its simplest form, suppose P is a (3,3)-corner polyhedron defined by $Ax \le b$, where

$$\gcd\,[A_S] = \gcd\,[A_S \mid b_S]$$

for all face 1-subsets and 2-subsets S. Then $P = P^1 = P^2$. We regard A as a non-singular linear transformation, and let $I(P)^*$ be the image of I(P) under A. Normal vectors and vertices of $I(P)^*$ are defined using obvious generalizations of definitions of their two-dimensional counterparts. We let $\bar{A}$ be the cofactor matrix of A. As before it can be shown that normal vectors of $I(P)^*$ are necessarily members of $M(\bar{A})^+$. However, it is unfortunately no longer true that normal vectors of $I(P)^*$ are necessarily atoms of the partially ordered set $(M(\bar{A})^+, \le)$. The following example serves as a counter-example.

Example 7-1 Let P be defined by

$$\begin{bmatrix} -4 & 1 & 1 \\ 3 & -5 & 4 \\ 3 & -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} .$$

It is easy to verify that

$$\gcd\ [A_S] = \gcd\ [A_S \mid b_S]$$

for all 1-subsets and 2-subsets S. The cofactor matrix $\bar{A}$ is given by

$$\bar{A} = \begin{bmatrix} 2 & 6 & 6 \\ -5 & -11 & -9 \\ 9 & 19 & 17 \end{bmatrix}.$$

Fig. 7-1 below illustrates $H(I(P)^*)$, determined by actual construction.

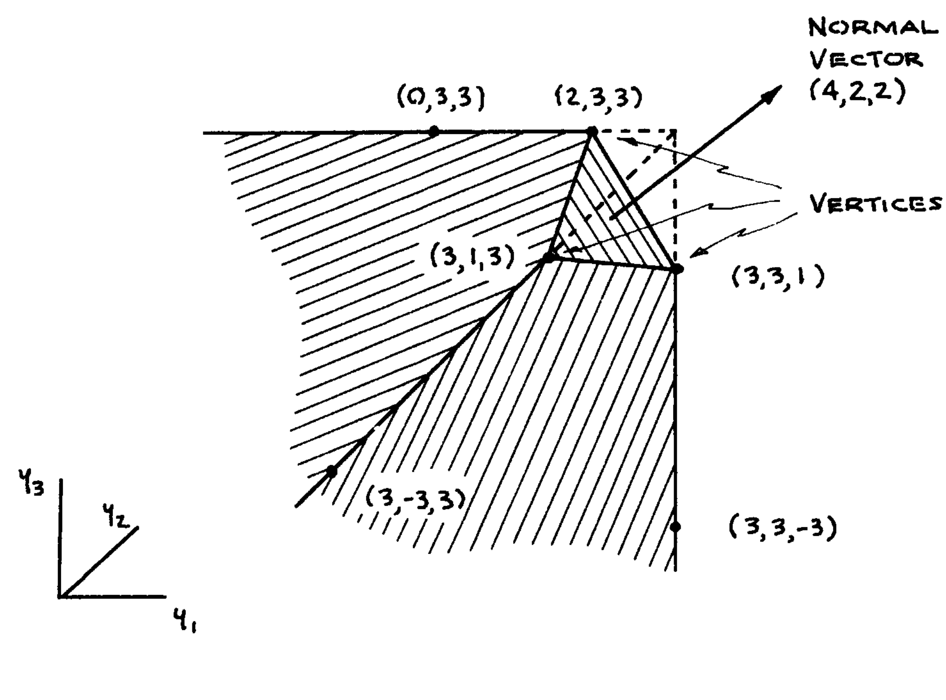


Fig. 7-1

The vector $\gamma = (4,2,2)$ is a normal vector of $I(P)^*$ at vertices $(3,1,3)$, $(3,3,1)$ and $(2,3,3)$. It is easily verified that $\gamma \in M(\bar{A})^+$. However it is also easily verified that $(4,0,0)$ and $(0,2,2)$ are members of $M(\bar{A})^+$. Thus the normal vector γ is not an atom of $(M(\bar{A})^+, \leq)$. □

Since normal vectors are not necessarily atoms, our two-dimensional integralization method does not generalize directly. In order to generalize the method it would first be necessary to identify some finite subset of $M(\bar{A})^+$ which contains all possible normal vectors of $I(P)^*$. Then a method for generating this finite subset of $M(\bar{A})^+$ would be required, along with a process for using this finite subset to generate all normal vectors and vertices of $I(P)^*$. In the two-dimensional case, it was the finite set of atoms of $(M(\bar{A})^+, \leq)$ that satisfied our first prerequisite. In the three-dimensional case, the appropriate finite subset of $M(\bar{A})^+$ may possibly be those members of $M(\bar{A})^+$ which occupy the first two ranks of the partially ordered set $(M(\bar{A})^+, \leq)$. This is merely a conjecture which seems to be a reasonable inductive inference to make from the two-dimensional case, once the set of atoms of $(M(\bar{A})^+, \leq)$ are found to be insufficient.

Our two-dimensional integralization method gives us reason to believe that there exist number-theoretic methods for integralizing (k,k)-polyhedra for arbitrary k. Our experience in seeking a generalization leads us to believe that inventing these methods is apt to be difficult. The problem remains a fascinating one for us, and one whose solution we believe would be of great value in the field of combinatorial optimization.

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UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Massachusetts Institute of Technology Project MAC	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP None

3. REPORT TITLE

Integral Convex Polyhedra and an Approach to Integralization

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
Sc.D. Thesis, Electrical Engineering Department, August 1970

5. AUTHOR(S) *(Last name, first name, initial)*

Edelberg, Murray

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1970	172	26

8a. CONTRACT OR GRANT NO. Nonr-4102(01) b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) MAC TR-74 9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. AVAILABILITY/LIMITATION NOTICES

Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	Advanced Research Projects Agency 3D-200 Pentagon Washington, D.C. 20301

13. ABSTRACT

Many combinatorial optimization problems may be formulated as integer linear programming problems, that is, problems of the form: given a convex polyhedron P contained in the non-negative orthant of n-dimensional space, find an integer point in P which maximizes (or minimizes) a given linear objective function. Well known linear programming methods would suffice to solve such a problem if: (i) P is an integral convex polyhedron, or (ii) P is transformed into the integral convex polyhedron that is the convex hull of the set of integer points in P, a process which is called integralization.

This thesis provides some theoretical results concerning integral convex polyhedra and the process of integralization. Necessary and sufficient conditions for a convex polyhedron P to have the integral property are derived in terms of the system of linear inequalities defining P. A number-theoretic method for integralizing two-dimensional convex polyhedra is developed which makes use of a generalization of the division theorem for integers. The method is applicable to a restricted class of higher dimensional polyhedra as well.

14. KEY WORDS

Integral Convex Polyhedra   Integer Convex Hulls   Integer Linear Programming
Combinatorial Optimization

DD FORM 1 NOV 65 1473 (M.I.T.)

UNCLASSIFIED
Security Classification