UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE G 2 - 98

TITLE	Poisson's Equation, Liebmann-Frankel Method (DOI Only)
TYPE	Open
NUMBER OF WORDS	Locations 6-64. Grid points and key words may fill memory.
TEMPORARY STORAGE	0-8
ACCURACY	10^{-3}
DURATION	4 PQ milliseconds, where P = number of interior points Q = number of times over grid For rectangular regions Q is roughly approximated by $\lvert 3/\log_{10} K^* \rvert$ where $K^* = 4\alpha_0 - 1$, $\alpha_0 = [2\sqrt{4-(\cos \pi/p + \cos \pi/q)^2}]/[\cos \pi/p + \cos \pi/q]^2$, $p + 1 = N$ = number of columns, $q + 1 = M$ = number of rows.

DESCRIPTION This code gives the solution V(x,y), corresponding to given boundary values of V and given interior values of p, of the difference equation $V_{\bar{x}x} + V_{\bar{y}y} = \rho h^2$ which approximates Poisson's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \rho(x,y).$$

The solution is found for all interior grid points of an arbitrary closed region with a square mesh of side h. The process used is that of Liebmann and Frankel, which requires an initial approximation (in this case 0) $V^0_{r,s}$ for each interior point (r,s) and then improves this approximation by the following iteration formula:

$$V_{r,s}^{n+1} = V_{r,s}^{n} + \alpha[V_{r-1,s}^{n+1} + V_{r+1,s}^{n} + V_{r,s-1}^{n+1} + V_{r,s+1}^{n} - 4\, V_{r,s}^{n}$$

$$- \rho_{r,s} h^2],$$

where $r = 1,2,\ldots,p-1$; $s = 1,2,\ldots,q-1$. For rectangular regions

$$\alpha = \alpha_0 = \frac{2 - \sqrt{4 - [\cos \pi/p + \cos \pi/q]^2}}{[\cos \pi/p + \cos \pi/q]^2}$$

and α_0 is computed by the "Determineα" routine, which also places it in location 5. This will suffice also for near-rectangular regions. For more peculiar regions omit "Determine α" routine and set N(5) = 1/4 before " Evaluate Interior Points".

Iteration stops when

$$\sum_{r,s} | V_{r,s}^{n+1} - V_{r,s}^{n} | \leq 5 \times 10^{-4}.$$

The maximum number of boundary and interior points depends on the complexity of the boundary but is limited to about 950 for very simple boundaries. First the boundary should be enclosed in a rectangular grid of M rows and N columns, preferably with $N \geq M$. Known boundary points, of course, should correspond to points of the grid. If a given interior point has a sequence position i in the M x N grid (reading from left to right and top to bottom) then its neighbors have the position i - N, i - 1, i + 1, i + N. Since the exterior points will not be stored it is necessary to find the actual addresses of the points in sequential position i, i-N, and i+N. This is done as follows: in preparing the problem a count is made of the number of points, with the sequences arranged in the same order as they are on the grid. These are called temporary key words and are placed on the tape after 00 65K as 20 nF 00F, 30 nF 00F, 40 nF 00F for interior, boundary, exterior sequences, respectively, where n is the number of points in a sequence. The total number of boundary and interior points plus the number of sequences of interior points

plus the number of sequences of interior points may not exceed 941. The tape of temporary key words must be terminated with 00F 00F. It is placed in the memory in locations (65ff) later occupied by the function values and the ρh^2's at the grid points. Temporary key words are operated on to form the addresses of grid points i, i-N, i+N, for each i which starts a sequence of interior points, and to place them into one word. These words are called the key words and are put into the memory in sequence starting at any location following the space required for the grid points. A suggested position is at address j = [998 minus number of sequences of interior points]. Key words are of the form $[m \times 2^{-9} + p \times 2^{-19} + q \times 2^{-39}]$, where N(m), N(p), N(q) are the (i - N)-th, i-th, (i + N)-th numbers of the grid, respectively.

All boundary values and ρh^2's are required to lie in the range $-1/5 < V, \rho h^2 < 1/5$. The V's and ρh^2's are written and punched on the input tape as K(+) or S(-) followed by up to six decimal digits and are put in sequential order except that the character J must precede each boundary sequence (i.e. V's) and N must precede each interior sequence (i.e. ρh^2's). The character F must be at the end of the input data. Function values V and the ρh^2's are stored by the "Input" routine in the same word as follows: The sign digit distinguishes boundary points (1) and interior points (0); V occupies the next 20 binary digits; ρh^2's comprises the last 19 digits. Hence only one memory location is needed for each grid point, although the accuracy has been correspondingly reduced.

Printing of the solution is done in sequence, with ρh^2 for each interior point on the same line as the V for that point.

"Input" parameters required are (1) N, the number of columns, to be placed in location 3 as $N \times 2^{-19}$, and (2) j, the address for the start of the key words, to be placed in location 4 as $j \times 2^{-39}$. "Determine α", parameters are (1) p = N - 1 in location 3 as $p \times 2^{-39}$,

and (2) q = M - 1 in location 4 as q x 2^{-39}. The "Print" parameter is s, the address of the last point to be printed, in location 4 as s x 2^{-39}. The first grid point is at location 56.

The complete routine requires the following tapes:

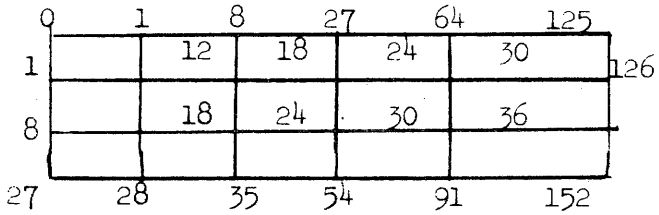
1. Decimal Order Input (X1)
2. Temporary key words (in 65ff) followed by 00F 00F and "Input" parameters.
3. Routine to form key words (in 10-64), followed by 26 10N.
4. Input routine (in 10 -41), followed by 26 10N.
5. Boundary points V and ρh^2's, followed by F
6. "Determine α" parameters and routine (in 5-46), followed by 22 14N.
7. Input parameters again and routine for evaluating interior point (in 6-54), followed by 26 6N.
8. Print parameter, print routine (in 10-25), and Print routine (P2) (in 26-43), followed by 26 10N.

The order listed must be adhered to although only 2,5 and parameters need to be constructed for each problem. For non-rectangular regions, instead of (6) set N(5) = 1/4. Code (4) overwrites (3), (6) overwrites (4) and (2), (7) overwrites (6), and (8) overwrites (7).

An example follows:

N = 7, p = 5,

M = 4, q = 3



all values times 10^{-4}

h = .01

For boundary points the given numbers are function values V; for interior points ρh^2's are given. All these numbers are in range. The temporary keys are 30 7F 00F, 20 4F 00F, 30 2F 00F, 20 4F 00F, 30 7F 00F, 00F 00F. The list of boundary values and ρh^2's is J, K, K0001, K0008, K0027, K0064, K0125, K0001, N, K0012, K0018, K0024, K003, J, K0126, K0008, N, K0018, K0024, K003, K0036, J, K0133, K0027, K0028, K0035, K0054, K0091, K0152,F.

Input parameters are N(3) = 6 x 2^{-19} and N(4) = [998-2] x 2^{-39} = 996 x 2^{-39}. "Determine α" routine parameters are N(3) = 5 x 2^{-39} and N(4) = 3 x 2^{-39}. The print parameter is N(4) = 79 x 2^{-39}.

RT: 4/10/59
DATE September 13, 1954
PROGRAMMED BY Richard F. King
APPROVED BY J. P. Nash

lgr

LOCATION	ORDER	NOTES	PAGE 1 G2
	FORM KEY WORDS (POISSON) 00 10K 10-64F		
0	41 7F		
	40 8F		N(7) = N(8) = 0
1	50 (65)F	From 46'	Q = Current temporary key
	01 3F		
2	L0 52L		Test for end of temporary keys
	36 4L		
3	41 F	By 41	Insert 0 at end of keys
	24 999F		Read in "Input"
4	L0 52L	From 2'	
	L0 52L		Test for interior sequence
5	36 47L	NO	
	41 5F	Yes	N(5) = N(6) = 0
6	40 6F		
	L5 7F		
7	L0 8F		N(0) = [number of interior points +
	26 8L		number of boundary points] x 2^{-19}
8	40 F		
	L5 1L		
9	46 26L		Plant address of current temporary
	46 42L		key
10	L0 51L		Plant address of previous temporary
	46 11L		key
11	50 F	By 10',17	
	01 3F	From 17'	Test for exterior sequence
12	L0 53L		
	36 18L	Yes	
13	11 3F	No	
	L5 5F		N(5) = [number of points we have
14	S4 F		backed up] x 2^{-19}
	40 5F		
15	L0 3F		
	32 20L	Yes	Have we backed up a row?
16	L5 11L	No	Plant address of previous temporary
	L0 51L		key.

LOCATION	ORDER		NOTES PAGE 2 G 2
17	46 11L		
	26 11L		
18	11 3F	from 12'	N(6) = [number of exterior points we have backed up] x 2^{-19}
	L5 6F		
19	S4 F		
	40 6F		
20	22 13L		
	L5 54L	From 15'	A = [address of first mesh word] x 2^{-19}
21	L4 F		
	40 1F		
22	L0 3F		N(1) m x 2^{-19} + p x 2^{-19}
	L4 6F		
23	00 10F		
	L4 1F		
24	40 1F		
	41 5F		
25	41 6F		N(5) = N(6) = 0
	26 26L	By 9	
26	50 F	From 32'	Test for exterior sequence
	01 3F		
27	L0 53L		
	36 33L	Yes	
28	11 3F	No	
	L5 5F	From 35	N(5) = [number of points we have progressed] x 2^{-19}
29	S4 F		
	40 5F		
30	L0 3F		Have we progressed a row?
	32 35L	Yes	
31	L5 26L	No	
	L4 51L		Plant address of next temporary
32	46 26L		key.
	26 26L		
33	11 3F	From 27'	N(6) = [number of exterior points
	L5 6F		we have progressed] x 2^{-19}
34	S4 F		

LOCATION	ORDER	NOTES	
	40 6F		
35	22 28L		
	L5 1F	From 30'	
36	L4 3F		$N(1) = m \times 2^{-9} + p \times 2^{-19} + q \times 2^{-39}$
	L0 6F		
37	10 20F		= key word
	42 1F		
38	L5 1F		
	26 39L		Store key word
39	40 S4	By 40'	
	L5 39L		
40	L4 51L		Plant addresses for next key word
	46 39L		
41	46 3L		
	26 42L		
42	50 F	By 9'	Q = [number of points in present
	00 3F		interior sequence] $\times 2^{-16}$
43	11 3F	From 48	
	L5 7F	From 50'	N(7) = [number of interior points
44	S4 F		+ number of exterior points +
	40 7F		number of boundary points] $\times 2^{-19}$
45	L5 1L		
	L4 51L		Set address for next temporary key
46	46 1L		
	26 1L		
47	L0 52L	From 5	Test for boundary sequence
	32 48L	NO	
48	26 43L	Yes	
	11 3F	From 47'	
49	L5 8F		N(8) = [number of exterior points]
	S4 F		$\times 2^{-19}$
50	40 8F		
	22 43L		
51	00 1F		
	00 F		2^{-19}

LOCATION	ORDER		NOTES PAGE 4 G 2
52	00 F		
	00 1F		2^{-39}
53	00 F		
	00 4F		4×2^{-39}
54	00 56F		
	00 F		56×2^{-19}
	26 10N		
INPUT	00 10K	10-41F	F - End of input N - Interior sequence J - Boundary Sequence K - Positive Number S - Negative Number
0	81 4F	From 18'	Read in next character
	22 15L		
1	75 1F	From 7	
	00 39F		
2	40 1F		
	50 28L		
3	L5 2F		
	74 F		
4	S5 F		
	40 F	From 21,22'	Code N3
5	81 4F		
	50 28L		
6	40 2F		
	L0 26L		
7	36 1L		
	L5 F		
8	66 1F		
	L5 24L		Test for interior point
9	36 23L	Yes	
	L5 29L	NO	
10	S4 F		
	10 1F		Round off and position boundary
11	32 12L		point number

LOCATION	ORDER		NOTES PAGE 5 G 2
	26 13L		
12	00 F		
	L4 26L		
13	40 (56)F	From 24'25'	
	L5 13L	By 14' 11'	Store point values and step
14	L4 31L		address of store order.
	46 31L		
15	L5 2F		
	L0 27L	From 0'	Test for F(end)
16	34 999F	Yes	
	00 38F		Test for K or S
17	36 19L	Yes	
	L1 24L		Set N(23) negative or positive
18	40 24L		
	26 L		Start next sequence
19	00 1F	From 17	Test for K
	32 21L		
20	L1 30L		$N(1) = -2^{-39}$ for S
	40 1F		
21	23 4L		Form number
	L5 30L	From 19'	$N(1) = 2^{-39}$ for K
22	40 1F		
	23 4L		Form number
23	S5 F	From 9	
	L4 31L		
24	10 21F		Roundoff and position
	36 13L		(interior point number) x h^2
25	L4 28L		
	26 13L		
26	80 F		
	00 10F		$-1 + 10 \times 2^{-39}$
27	00 F		
	00 14F		14×2^{-39}
28	00 F		
	00 10F		10×2^{-39}

LOCATION	ORDER	NOTES
29	00 F	
	80 F	2^{-20}
30	00 F	
	00 1F	2^{-39}
31	00 1F	
	00 F	2^{-19}
	26 10N	
N(3) = p x 2^{-39}		
N(4) = q x 2^{-39}		Determine α
00 5K	5-15F	
0	51 10L	
	00 2F	A Q = π x 2^{-39}
1	22 1L	
	66 3F	A = π/p
2	85 F	
	50 2L	N(3) = (cos π/p) 1/2
3	26 16F	
	L5 2F	
4	40 3F	
	51 10L	
5	00 2F	
	66 4F	A = π/q
6	85 F	
	50 6L	
7	26 16F	A = (cos π/p) 1/2
	L5 2F	
8	L4 3F	N(3) = (cos π/p + cos π/q) 1/2
	40 3F	
9	26 999F	End of interlude
	50 3F	
10	40F 00 2853	
	9816 3397 J	π/4
	00 16K	
Sine, Cosine (T1) 16 - 46F		
	26 5N	

LOCATION	ORDER		NOTES PAGE 7 G 2
0015K	15-22F		
0	7J 3F		$N(3) = \left(\frac{\cos \pi/p + \cos \pi/q}{2}\right)^2$
	40 3F		
1	L5 7L		
	L0 3F		
2	10 2F		
	50 2L		$A = 1/4 \left[1 - \left(\frac{\cos \pi/p + \cos \pi/q}{2}\right)^2\right]$
3	26 23F		
	49 1F		
4	L0 2F		
	66 3F		$N(5) = \alpha$
5	S5 F		
	40 5F		$= \frac{1/2 \sqrt{1/4 \left[1 - \left(\frac{\cos \pi/p + \cos \pi/q}{2}\right)^2\right]}}{\left[(\cos \pi/p + \cos \pi/q)\ 1/2\right]^2}$
6	24 999F		
	0F F		Read in next routine
7	7L 4095F		Stop
	LL 4095F		$1 - 2^{-39}$
00 23K	23 - 32F		Square Root Routine (R1)
	22 14N		
Evaluate Interior Points (Poisson)			$N(5) = \alpha$
00 6K	6- 54F		
0	41 3F		$N(3) = 0$
	L3 S4	By 9',39	Test for end of grid
1	32 35L		
	L5 S4	By 10, 39'	$N(2)$ = current key word
2	40 2F	From 35	
	46 7L		
3	46 30L		Plant p,q
	42 16L		
4	L4 41L		Plant p+1
	46 15L		
5	L0 42L		Plant p-1
	46 18L		
6	10 30F		
	42 19L		Plant m

LOCATION	ORDER		NOTES PAGE 8 G 2
7	L5 (p)F	By 2'	
	40 4F		N(4) A = current function value
8	36 11L		Test for boundary point
	L5 L	Yes	
9	L4 41L		Plant key word addresses
	42 L		
10	42 1L		
	22 L		End of interior sequence
11	50 46L	From 8	
	00 21F		$A = 1/2\ (\rho\ h^2/4)$
12	10 3F		
	50 4F		
13	J0 47L		$A = N(0) = 1/2\ (+\ 4\ \phi^n + \rho h^2)$
	S4 F		
14	00 2F		
	40 F		
15	50 (p+1)F	By 4'	
	J0 47L		
16	S5 F		
	50 (q)F	By 3'	
17	J0 47L		
	S4 F		$A = 1/2\ (4\ \phi^{-n})$
18	50 (p-1)F	By 5'	
	J0 47L		
19	S4 F		
	50 (m)F	By 6'	
20	J0 47L		
	S4 F		
21	L0 F		
	00 1F		$N(0) = 4\phi^{-n} - 4\phi^n - \rho h^2$
22	40 F		
	50 F		
23	75 5F		$N(0) = \alpha[4\phi^{-n} - 4\phi^n - \rho h^2]$
	L4 45L		
24	40 F		

LOCATION	ORDER		NOTES PAGE 9 G 2
	10 1F		
25	50 4F		$N(1) = 1/2 \left\{ \phi^n + \alpha[11] \right\} = 1/2\ \phi^{n+1}$
	J0 47L		
26	S4 F		
	40 1F		
27	50 4F		
	J0 46L		
28	S5 F		
	50 1F		$N(p) = 1/2\ [\phi^{n+1} + 2^{-20} \times \rho\, h^2]$
29	J0 47L		
	S4 F		
30	40 (p)F	By 3	
	50 F		
31	J0 48L		$N(3) = \Sigma\ \lvert N_{i+1}(p) - N_i(p) \rvert$
	S7 F		
32	L4 3F		
	40 3F		
33	36 34L		Test for overflow [N(43) < 0]
	40 43L	Yes	
34	L5 2F	From 33	
	L4 41L		Step m, p, q by 1
35	26 2L		
	L1 43L	From 1	Test for overflow
36	36 38L		
	L5 44L	NO	
37	L0 3F		Test for end
	34 999F		
38	49 43L	From 36	Reset $N(43) \geq 0$
	L5 40L		
39	42 L		Reset addresses and start over on
	42 1L		grid
40	26 L		
	0F S4		Address used
41	00 1025F		
	00 1025F		$2^{-9} + 2^{-19} + 2^{-29} + 2^{-39}$

LOCATION	ORDER		NOTES PAGE 10 G 2
42	00 1026F		
	00 1025F		$2^{-9} + 2 \times 2^{-19} + 2^{-29} + 2^{-39}$
43	40 F		
	00 F		1/2
44	00F 00 0005		
	0000 0000 J		5×10^{-4}
45	00 F		
	80 F		2^{-20}
46	00 F		
	7L 4095F		00 -- 0 | 001 -- 1
47	7L 4095F		
	80 F		011-- 1 | 100 -- 0
48	LL 4095F		
	00 F		11 -- 1 | 00 -- 0
	26 6N		
Print (Poisson)00 10K			10-25F
0	92 129F	From 11	
	50 (56)F	By 9'	
1	J0 14L		Extract, position + print function
	00 40F		
2	50 3F		values
	50 2L		
3	26 26F		
	L5 (56)F	By 10	
4	36 5L	NO	Test for boundary point
	22 8L	Yes	
5	22 5L	From 4	
	50 (56)F	By 10'	
6	J0 15L		Extract, position, and print ρh^2's
	00 60F		
7	50 3F		
	50 7L		
8	26 26F		
	19 38F	From 4'	
9	L4 L		Set addresses for next point

LOCATION	ORDER		NOTES PAGE 11 G 2
	42 L		
10	42 3L		
	42 5L		
11	L0 13L		Test for end
	32 12L	Yes	
12	26 L	NO	Print values for next point
	0F F		Final stop
13	92 129F		
	50 1S4		End constant
14	7L 4095F		
	80 F		011 -- 1 | 100 -- 0
15	00 F		
	7L 4095F		000 -- - | -11 -- 1
	00 26K		
	Print Routine (P2)		
	26 10N		